

Grassmannian Geometry of Scattering Amplitudes

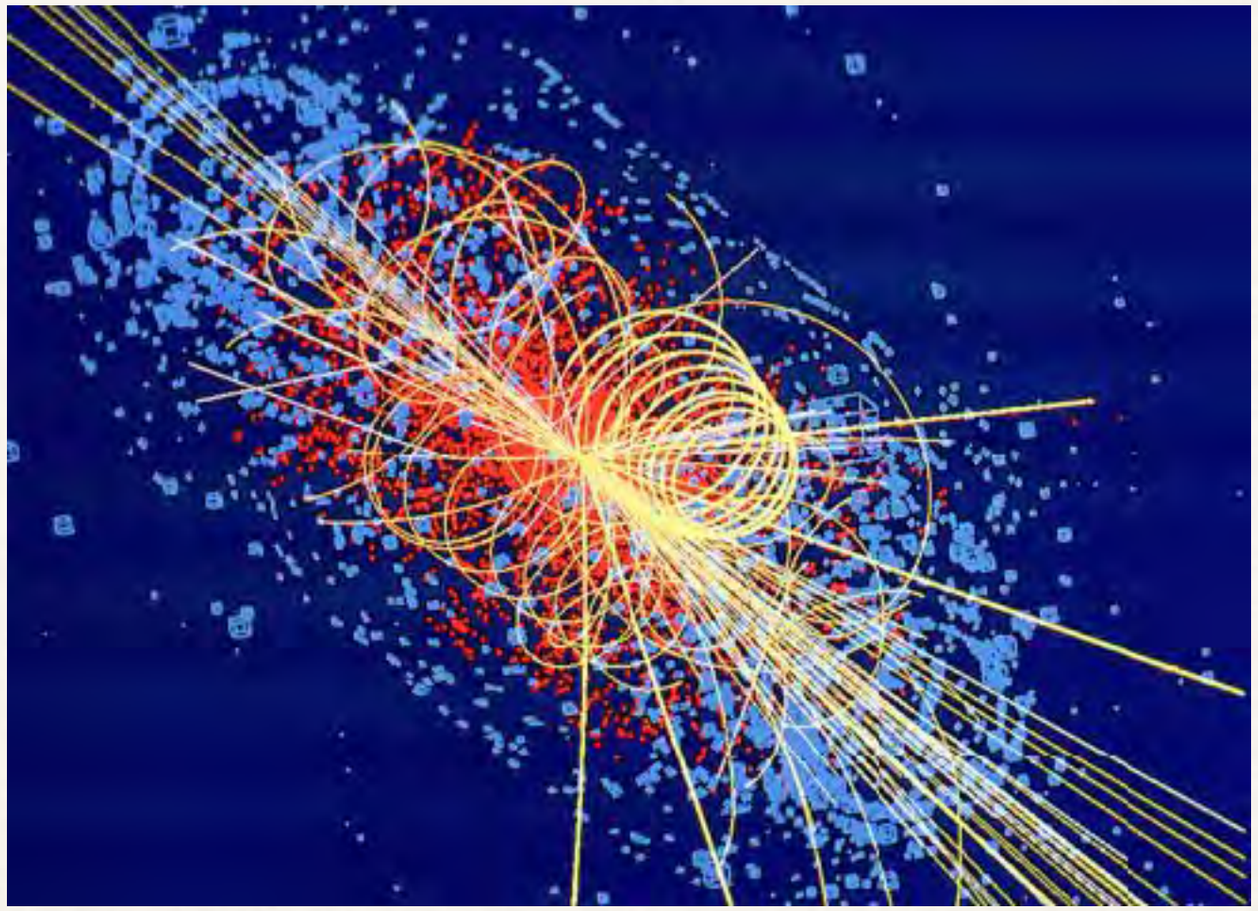
LECTURE 1

Jaroslav Trnka

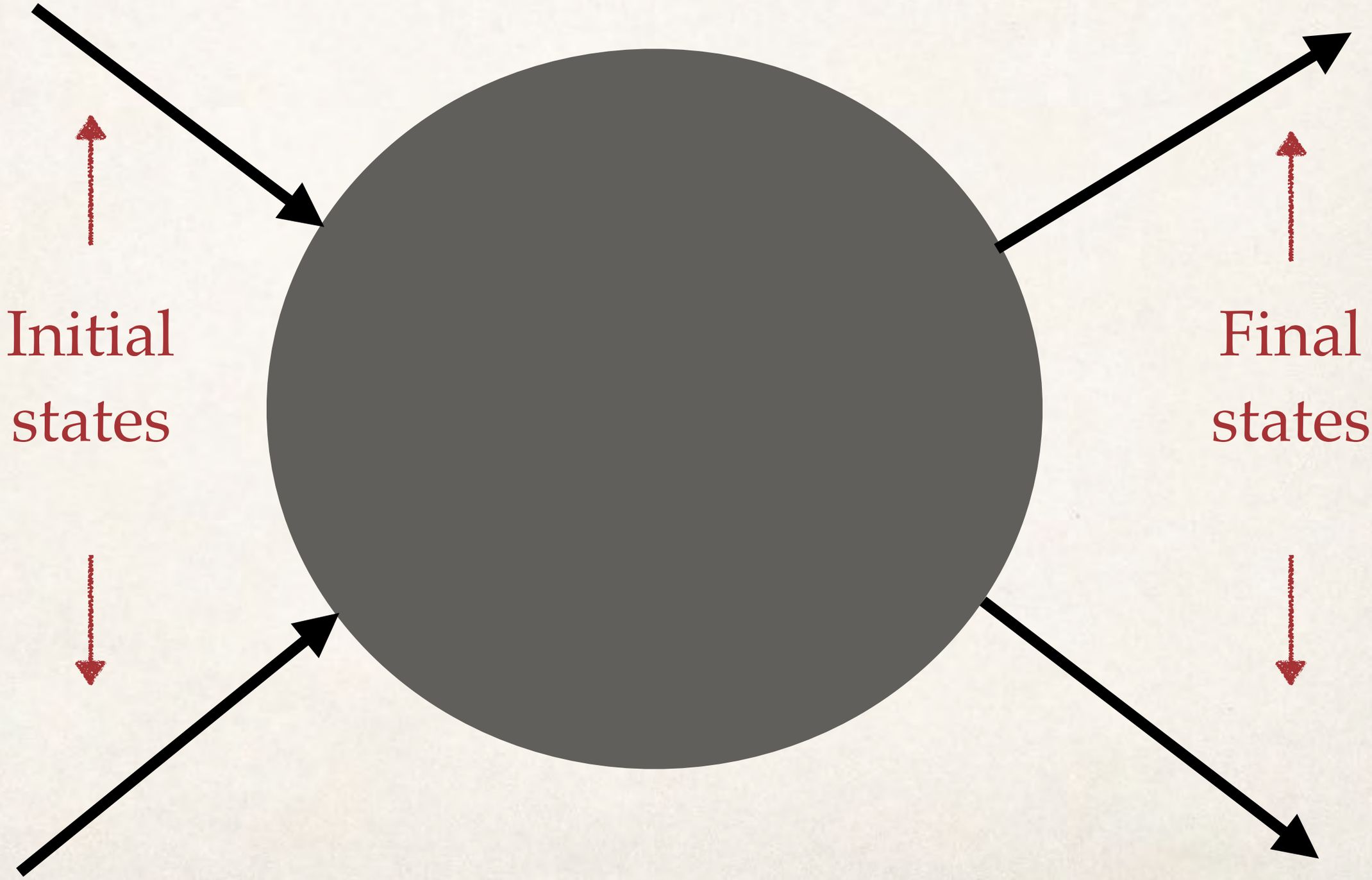
Center for Quantum Mathematics and Physics (QMAP), UC Davis

Amplitudes School, Edinburgh, July 2017

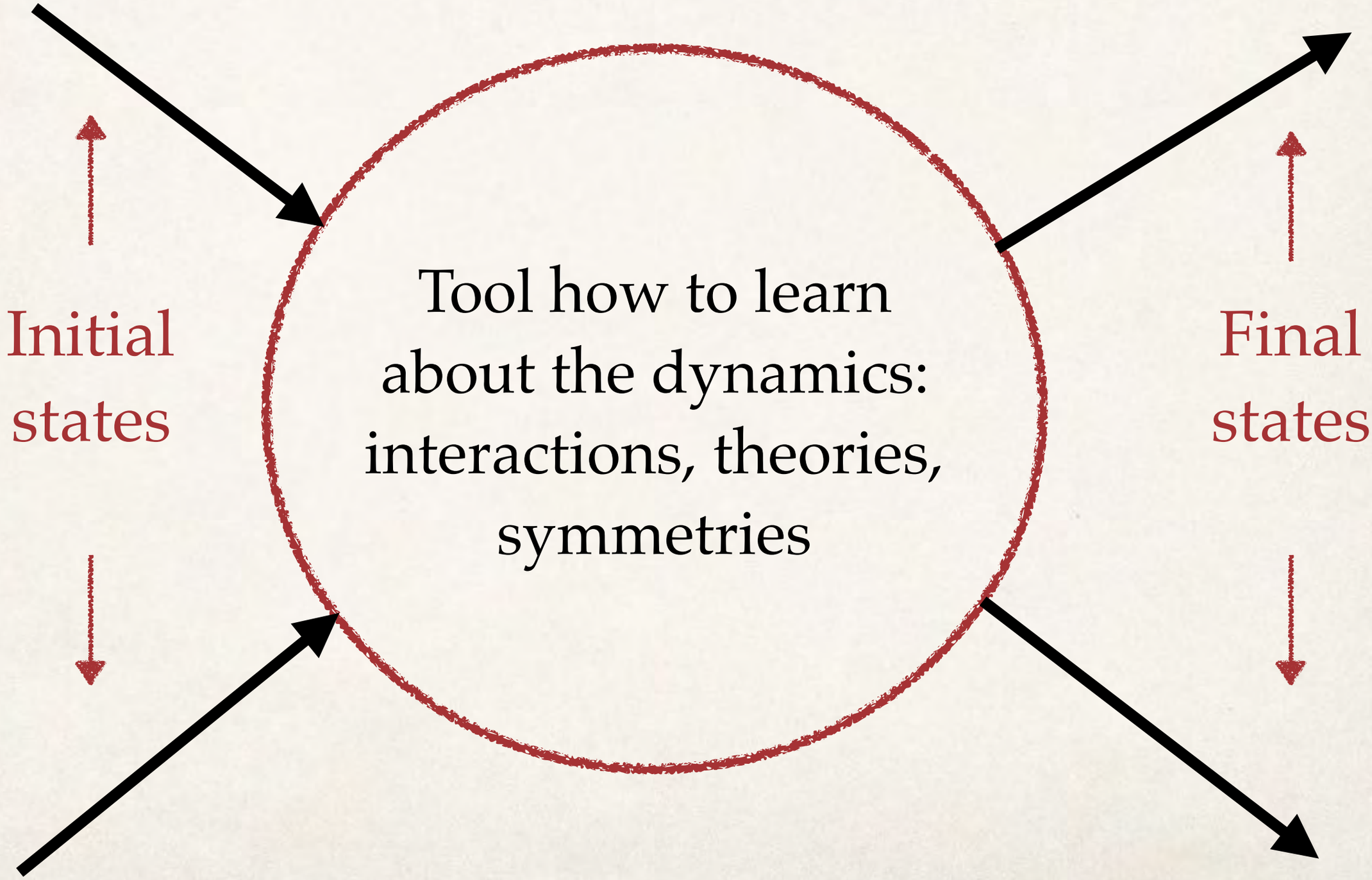
Particle experiments:
our probe to fundamental laws of Nature



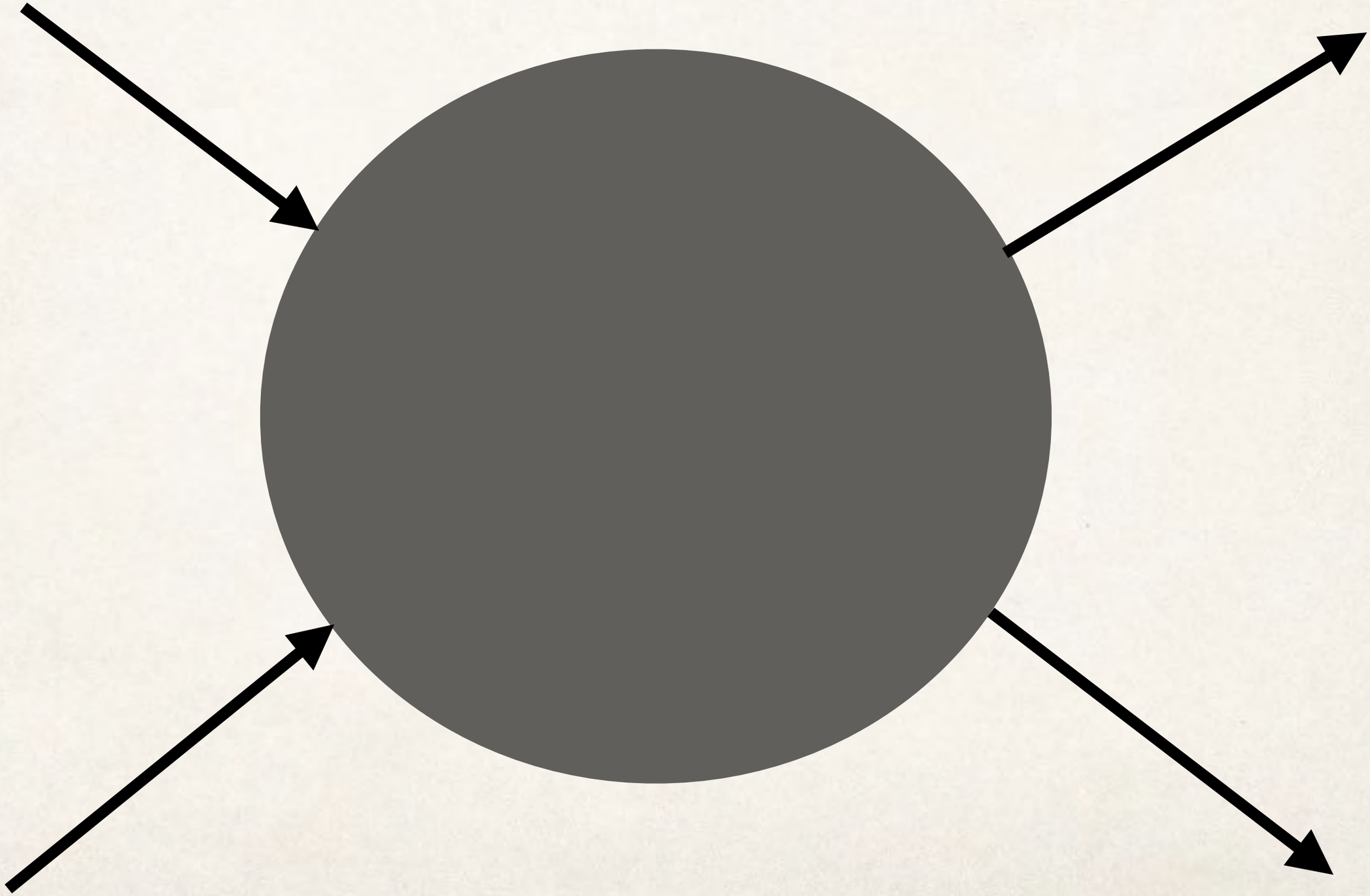
Theorist's perspective: scattering amplitude



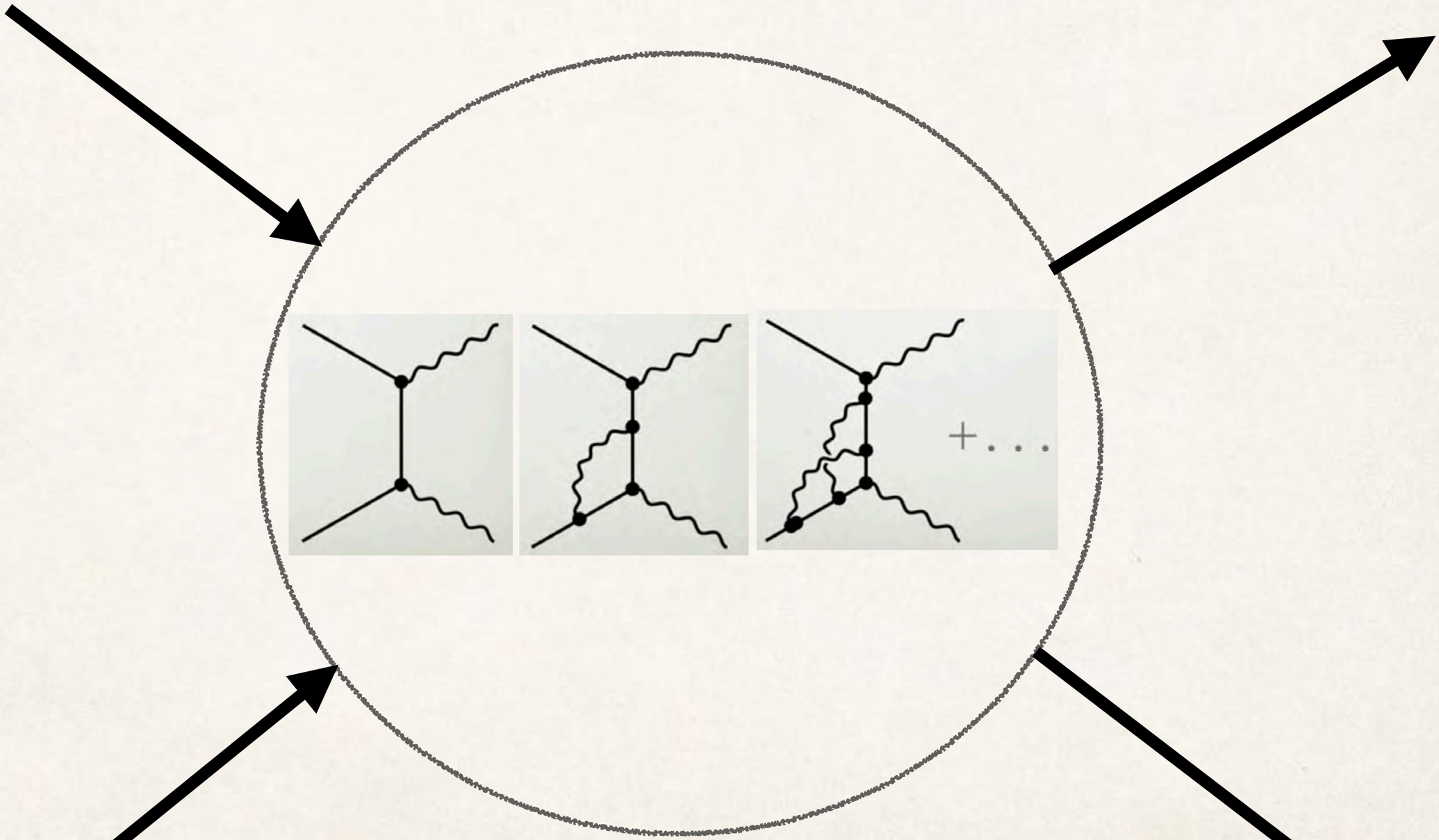
Phenomenology



What does the blob really represent?

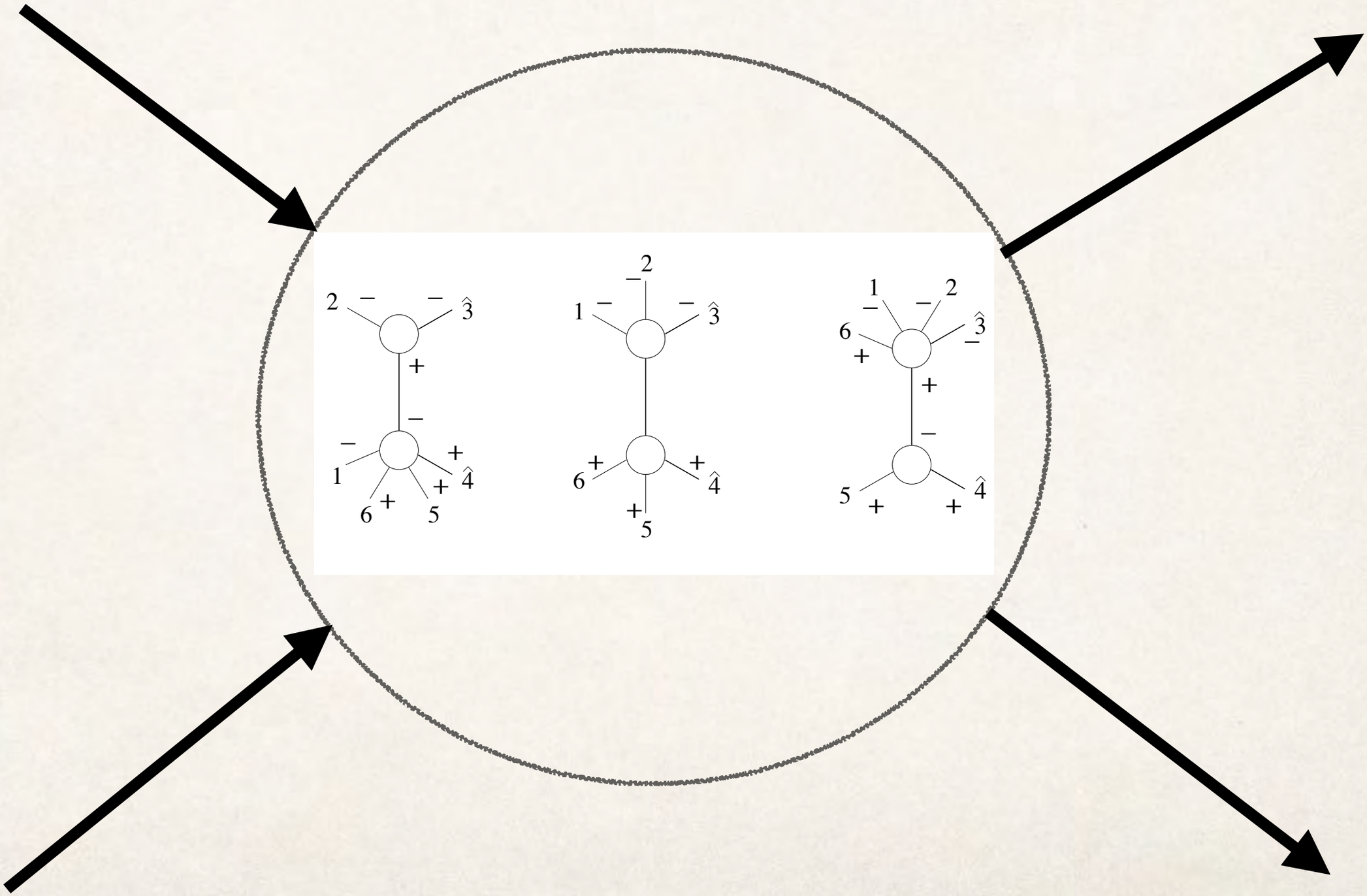


What does the blob really represent?

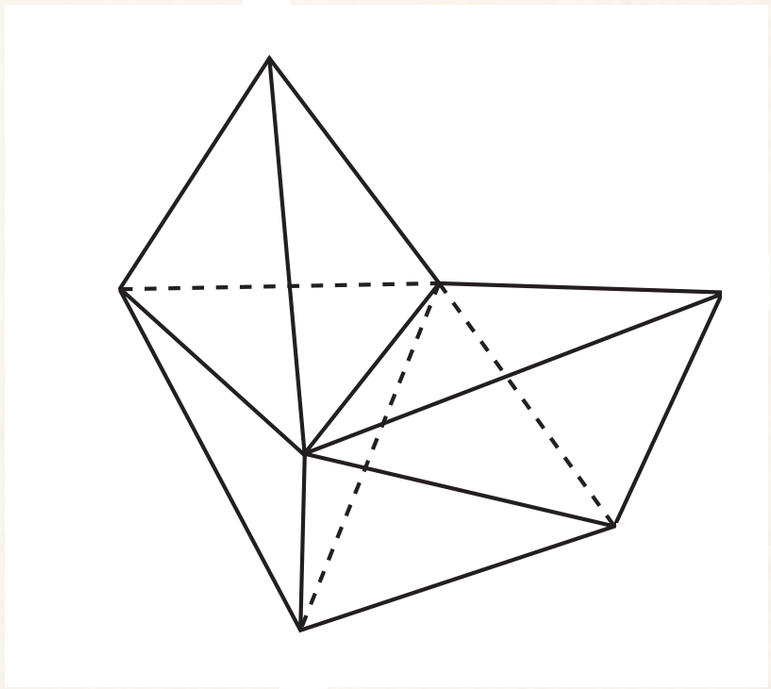


but there is more than that…..

It can be for example
a sum of different pictures



And in a special case
even something more surprising



Overview of lectures

- Lecture 1: Review of scattering amplitudes
- Lecture 2: On-shell diagrams
- Lecture 3: Amplituhedron
- Lecture 4: Beyond planar limit

Motivation

Perturbative QFT

(Dirac, Heisenberg, Pauli; Feynman, Dyson, Schwinger)

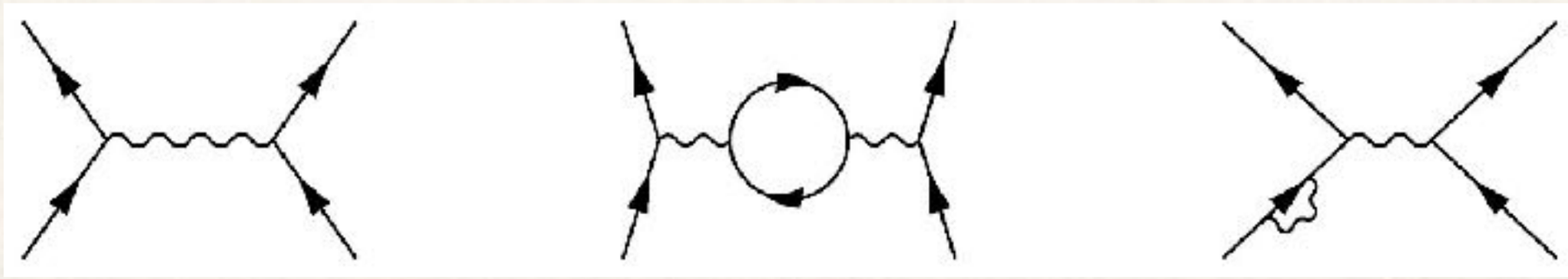


- Fields, Lagrangian, Path integral

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\overline{\psi} \not{D} \psi - m\overline{\psi}\psi \qquad \int \mathcal{D}A\, \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\; e^{iS(A,\psi,\overline{\psi},J)}$$

- Feynman diagrams: pictures of particle interactions

  Perturbative expansion: trees, loops



Dualities

- At strong coupling: perturbative expansion breaks

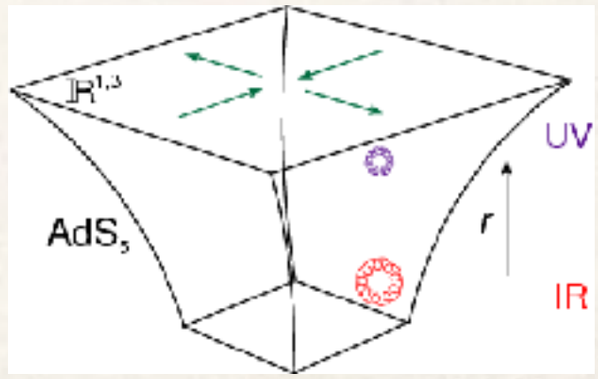


- Surprises: dual to weakly coupled theory
  - Gauge-gauge dualities

    (Montonen-Olive 1977, Seiberg-Witten 1994)

  - Gauge-gravity duality

    (Maldacena 1997)



Incomplete picture

- Our picture of QFT is incomplete

- Also, tension with gravity and cosmology

> If there is a new way of thinking about QFT,
> it must be seen even at weak coupling

- Explicit evidence: scattering amplitudes

Colliders at high energies

- Proton scattering at high energies

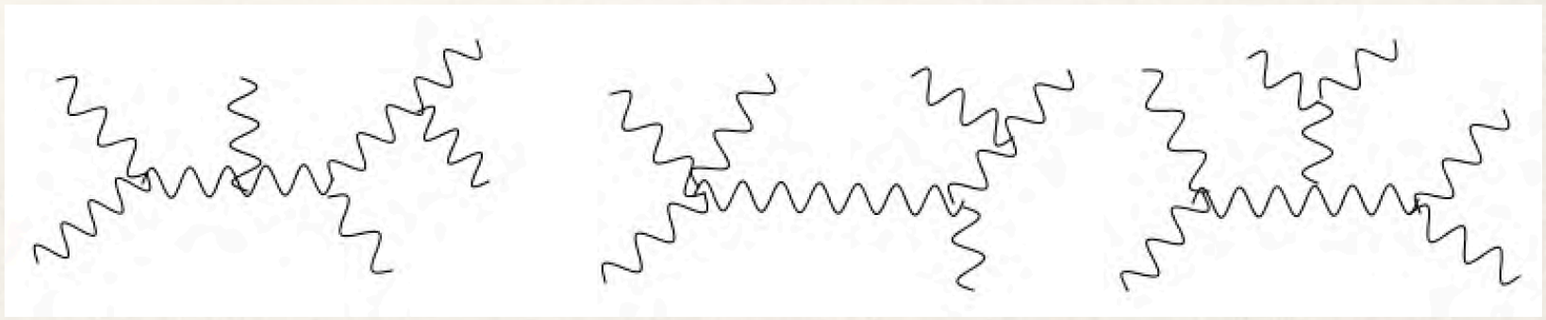




LHC - gluonic factory

- Needed: amplitudes of gluons for higher multiplicities

$$gg \to gg \ldots g$$



Early 80s

- Status of the art: $gg \to ggg$

Brute force calculation

24 pages of result



$$(k_1 \cdot k_4)(\epsilon_2 \cdot k_1)(\epsilon_1 \cdot \epsilon_3)(\epsilon_4 \cdot \epsilon_5)$$

New collider

- 1983: Superconducting Super Collider approved
- Energy 40 TeV: many gluons!





- Demand for calculations, next on the list: $gg \to gggg$

Parke-Taylor formula

(Parke, Taylor 1985)





- Process $gg \to gggg$
- 220 Feynman diagrams, $\sim$100 pages of calculations
- 1985: Paper with 14 pages of result

GLUONIC TWO GOES TO FOUR

Stephen J. Parke and T.R. Taylor
Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, IL 60510
U.S.A.

ABSTRACT

The cross section for two gluon to four gluon scattering is given in a form suitable for fast numerical calculations.

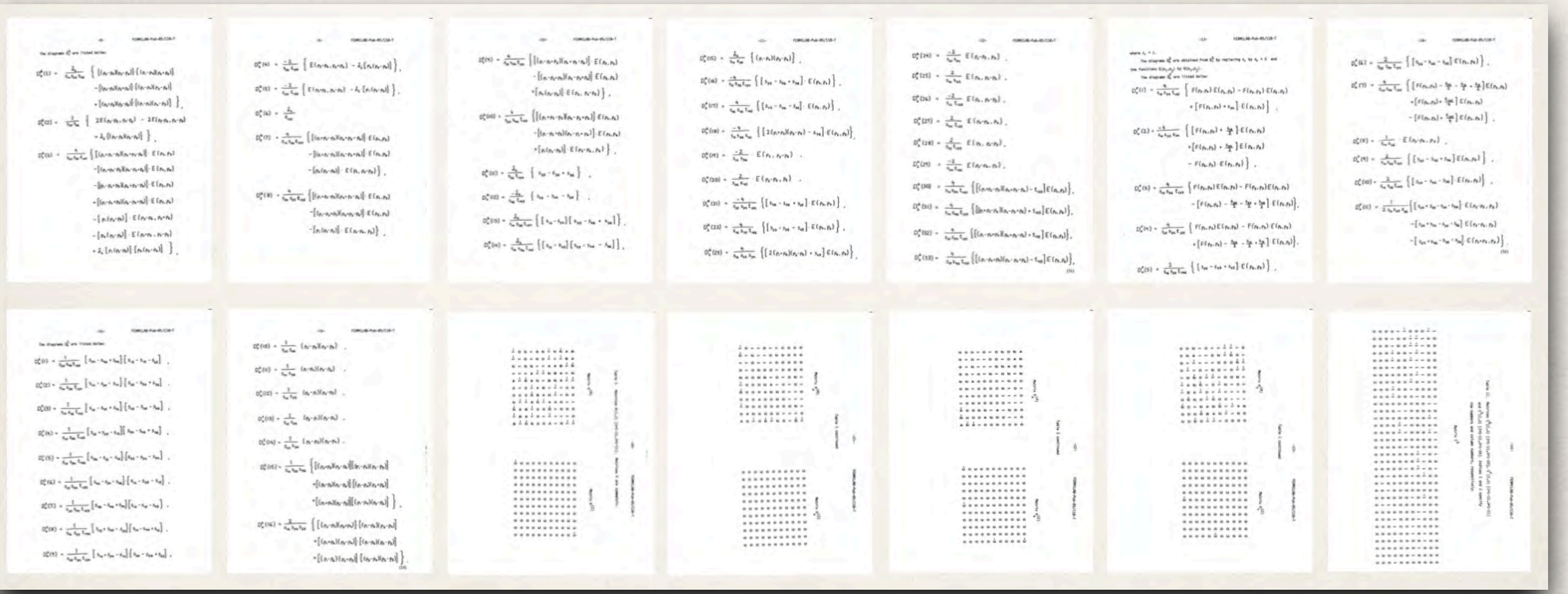
Parke-Taylor formula





- Process $gg \to gggg$

- 220 Feynman diagrams, $\sim$100 pages of calculations



Parke-Taylor formula





Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

Parke-Taylor formula





Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

- Within a year they realized

$$\mathcal{M}_6 = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

Spinor-helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}}\lambda_a\tilde{\lambda}_{\dot{a}}$$
$$\langle 12\rangle = \epsilon_{ab}\lambda^{(1)}_a\lambda^{(2)}_b$$
$$[12] = \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}^{(1)}_{\dot{a}}\tilde{\lambda}^{(2)}_{\dot{b}}$$

(Mangano, Parke, Xu 1987)

Parke-Taylor formula





Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

- Within a year they realized

$$\mathcal{M}_n = \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \ldots \langle n1 \rangle}$$

AN AMPLITUDE FOR n GLUON SCATTERING

STEPHEN J. PARKE and T. R. TAYLOR

Fermi National Accelerator Laboratory

P.O. Box 500, Batavia, IL 60510.

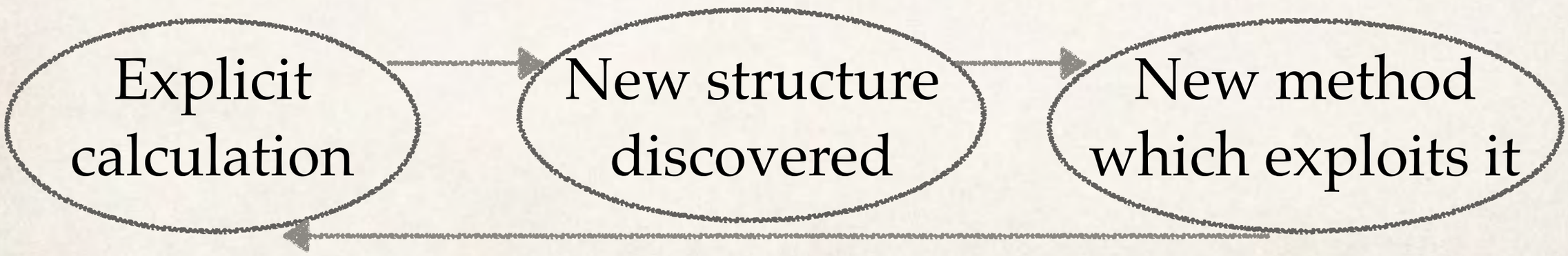
Birth of amplitudes

- New field in theoretical particle physics

New methods and effective calculations

Uncovering new structures in QFT

"Road map"



Kinematics of massless particles

Massless particles

- Parameters of elementary particles of spin S
  - Spin $s = (-S, S)$
  - Mass m
  - Momentum p^μ

On-shell (physical) particle

$$p^2 = m^2$$

- Massless particle: $m = 0 \quad p^2 = 0$

spin = helicity: only two extreme values $h = \{-S, S\}$

Example: photon $h = (+, -)$

$s = 0$ missing

Polarization vectors

- Spin 1 particle is described by vector ϵ^μ

2 degrees of freedom

4 degrees of freedom

- Null condition: $\epsilon \cdot \epsilon^* = 0$ 3 degrees of freedom left

- We further impose: $\epsilon \cdot p = 0$

Identification

$\epsilon_\mu \sim \epsilon_\mu + \alpha p_\mu$

Feynman diagrams depend on α

gauge dependence

Spinor helicity variables

- Standard SO(3,1) notation for momentum

$$p^\mu = (p_0, p_1, p_2, p_3) \qquad p_j \in \mathbb{R}$$

$$p^2 = p_0^2 + p_1^2 + p_2^2 - p_3^2$$

- We use SL(2,C) representation

$$p_{ab} = \sigma^\mu_{ab} p_\mu = \begin{pmatrix} p_0 + ip_1 & p_2 + p_3 \\ p_2 - p_3 & p_0 - ip_1 \end{pmatrix}$$

On-shell: $p^2 = \det(p_{ab}) = 0$

$$\boxed{\text{Rank}\,(p_{ab}) = 1}$$

Spinor helicity variables

- We can then write $p_{ab} = \lambda_a \kappa_b$

- SL(2,C): dotted notation $p_{a\dot{b}} = \lambda_a \widetilde{\lambda}_{\dot{b}}$

  $\widetilde{\lambda}$ is complex conjugate of λ

- Little group transformation

$$\lambda \to t\lambda \qquad \widetilde{\lambda} \to \frac{1}{t}\widetilde{\lambda}$$

leaves momentum unchanged $\quad p \to p$

3 degrees of freedom

Spinor helicity variables

- Momentum invariant $\quad (p_1 + p_2)^2 = (p_1 \cdot p_2)$

$$p_1^\mu = \sigma^\mu_{a\dot{a}}\, \lambda_{1a} \widetilde{\lambda}_{1\dot{a}} \qquad p_2^\mu = \sigma^\mu_{b\dot{b}}\, \lambda_{2b} \widetilde{\lambda}_{2\dot{b}}$$

- Plugging for momenta

$$(p_1 \cdot p_2) = (\sigma^\mu_{a\dot{a}} \sigma_{\mu\, b\dot{b}})\, (\lambda_{1a}\lambda_{2b})(\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}})$$

$$\epsilon_{ab}\, \epsilon_{\dot{a}\dot{b}} \qquad = (\epsilon_{ab}\lambda_{1a}\lambda_{2b})(\epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}})$$

Define: $\quad \langle 12 \rangle \equiv \epsilon_{ab}\lambda_{1a}\lambda_{2b} \qquad [12] \equiv \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}_{1\dot{a}}\widetilde{\lambda}_{2\dot{b}}$

Invariant products

- Momentum invariant

$$s_{ij} = (p_i + p_j)^2 = \langle ij \rangle [ij]$$

Angle brackets $\langle ij \rangle = \epsilon_{ab} \lambda_{ia} \lambda_{jb}$

Square brackets $[ij] = \epsilon_{\dot{a}\dot{b}} \lambda_{i\dot{a}} \lambda_{j\dot{b}}$

- Antisymmetry

$$\langle 21 \rangle = -\langle 12 \rangle \qquad [21] = -[12]$$

- More momenta

$$(p_1 + p_2 + p_3)^2 = \langle 12 \rangle [12] + \langle 23 \rangle [23] + \langle 13 \rangle [13]$$

Invariant products

- Shouten identity

$$\langle 13\rangle\langle 24\rangle = \langle 12\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle$$

- Mixed brackets

$$\langle 1|2+3|4] \equiv \langle 12\rangle[24] + \langle 13\rangle[34]$$

- Momentum conservation

$$\sum_{i=1}^{n} \lambda_{ia}\widetilde{\lambda}_{i\dot{a}} = 0$$

Non-trivial conditions:

Quadratic relation between components

Polarization vectors

- Two polarization vectors

$$\epsilon_+^\mu = \sigma^\mu_{a\dot{a}} \frac{\eta_a \widetilde{\lambda}_{\dot{a}}}{\langle \eta \lambda \rangle} \qquad \epsilon_-^\mu = \sigma^\mu_{a\dot{a}} \frac{\lambda_a \widetilde{\eta}_{\dot{a}}}{[\widetilde{\eta}\, \widetilde{\lambda}]}$$

where $\eta, \widetilde{\eta}$ are auxiliary spinors

Note that $(\epsilon_+ \cdot \epsilon_-) = 1$

- Freedom in choice of $\eta, \widetilde{\eta}$ corresponds to

$$\epsilon^\mu \sim \epsilon^\mu + \alpha\, p^\mu$$

- Gauge redundancy of Feynman diagrams

Scaling of amplitudes

- Consider some amplitude $A(- + - - + \cdots -)$

$$A = (\epsilon_1 \epsilon_2 \ldots \epsilon_n) \cdot Q$$

depends only on momenta

- Little group scaling

$$\lambda \to t\lambda \qquad \tilde{\lambda} \to \frac{1}{t}\tilde{\lambda}$$

$$\longrightarrow \quad p \to p \qquad \epsilon_+ \to \frac{1}{t^2} \cdot \epsilon_+ \qquad \epsilon_- \to t^2 \cdot \epsilon_- \quad \longrightarrow$$

$$A(i^-) \to t^2 \cdot A(i^-)$$

$$A(i^+) \to \frac{1}{t^2} \cdot A(i^+)$$

Back to Parke-Taylor formula

- Let us consider $A(1^-2^-3^+4^+5^+6^+)$
- Scaling $A\left(t\lambda_i, \frac{1}{t}\widetilde{\lambda}_i\right) = t^2 \cdot A(\lambda_i, \widetilde{\lambda}_i)$ for particles 1,2

$$A\left(t\lambda_i, \frac{1}{t}\widetilde{\lambda}_i\right) = \frac{1}{t^2} \cdot A(\lambda_i, \widetilde{\lambda}_i) \quad \text{for particles } 3,4,5,6$$

- Check for explicit expression

$$A_6 = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

If only $\langle ij\rangle$ allowed
the form is unique

Helicity amplitudes

- In Yang-Mills theory we have + or - "gluons"

$$A_6(1^-2^-3^+4^+5^+6^+)$$

- We denote k: number of - helicity gluons

- Some amplitudes are zero

$$A_n(+++\cdots+) = 0$$
$$A_n(-++\cdots+) = 0$$
$$A_n(---\cdots-) = 0$$
$$A_n(+--\cdots-) = 0$$

First non-trivial: k=2

$$A_n(--+\cdots+)$$

Parke-Taylor formula for tree level

Three point amplitudes

Three point kinematics

- Gauge invariant building blocks: on-shell amplitudes

$$p_1^2 = p_2^2 = p_3^2 = 0 \qquad\qquad p_1 + p_2 + p_3 = 0$$

- Plugging second equation into the first

$$(p_1 + p_2)^2 = (p_1 \cdot p_2) = 0$$

- Similarly we get for other pairs

$$(p_1 \cdot p_2) = (p_1 \cdot p_3) = (p_2 \cdot p_3) = 0$$

These momenta are very constrained!

Three point kinematics

- Use spinor helicity variables trivializes on-shell condition

$$p_1 = \lambda_1\widetilde{\lambda}_1, \quad p_2 = \lambda_2\widetilde{\lambda}_2, \quad p_3 = \lambda_3\widetilde{\lambda}_3$$

- The mutual conditions then translate to

$$(p_1 \cdot p_2) = \langle 12\rangle[12] = 0$$

- And similarly for other two pairs

$$(p_1 \cdot p_3) = \langle 13\rangle[13] = 0 \qquad (p_2 \cdot p_3) = \langle 23\rangle[23] = 0$$

Two solutions

- We want to solve conditions

$$\langle 12\rangle[12] = \langle 13\rangle[13] = \langle 23\rangle[23] = 0$$

- Solution 1: $\langle 12\rangle = 0$ which implies $\lambda_2 = \alpha\lambda_1$

  Then we also have $\langle 23\rangle = \alpha\langle 13\rangle$

  And we set $\langle 13\rangle = 0$ by demanding $\lambda_3 = \beta\lambda_1$

$$\boxed{\lambda_1 \sim \lambda_2 \sim \lambda_3}$$

Two solutions

- Solution 2: $[12] = [23] = [13] = 0$

$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

- Let us take this solution

$$p_1 = \lambda_1 \widetilde{\lambda}_1, \quad p_2 = \alpha \lambda_2 \widetilde{\lambda}_1, \quad p_3 = (-\lambda_1 - \alpha \lambda_2) \widetilde{\lambda}_1$$

complex momenta

No solution for real momenta

Three point amplitudes

- Gauge theory: scattering of three gluons (not real)
- Building blocks: $\langle 12\rangle, \langle 23\rangle, \langle 13\rangle, [12], [23], [13]$
- Mass dimension: each term $\sim m$
- Three point amplitude $A_3 \sim \epsilon^3 p \sim p \sim m$

Three point amplitudes

- Two options

$$A_3^{(1)} = \langle 12\rangle^{a_1}\langle 13\rangle^{a_2}\langle 23\rangle^{a_3} \qquad A_3^{(2)} = [12]^{b_1}[13]^{b_2}[23]^{b_3}$$

- Apply to $A_3(1^-, 2^-, 3^+)$

$$\begin{aligned} A_3(t\lambda_1, t^{-1}\widetilde{\lambda}_1) &= t^{a_1+a_2}\cdot A_3 \qquad a_1+a_2=2 \qquad a_1=3 \\ A_3(t\lambda_2, t^{-1}\widetilde{\lambda}_2) &= t^{a_1+a_3}\cdot A_3 \longrightarrow a_1+a_3=2 \longrightarrow a_2=-1 \\ A_3(t\lambda_3, t^{-1}\widetilde{\lambda}_3) &= t^{a_2+a_3}\cdot A_3 \qquad a_2+a_3=-2 \qquad a_3=-1 \end{aligned}$$

Three point amplitudes

- Similarly for $A_3(1^+, 2^+, 3^-)$

- Two fundamental amplitudes

$$A_3(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

$$A_3(1^+, 2^+, 3^-) = \frac{[12]^3}{[13][23]}$$

This is true to all orders: just kinematics

Three point amplitudes

- Collect all $(--+)$ amplitudes

$$A_3(1^-,2^-,3^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle}$$

$$A_3(1^-,2^+,3^-) = \frac{\langle 13\rangle^3}{\langle 12\rangle\langle 23\rangle}$$

$$A_3(1^+,2^-,3^-) = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 13\rangle}$$

$$\frac{\langle a\,b\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

where a,b are - helicity gluons

- Similarly for $(++-)$ amplitudes

$$\frac{[ab]^4}{[12][23][31]}$$

where a,b are + helicity gluons

Three point amplitudes

- Using similar analysis we find for gravity

$$\frac{\langle ab\rangle^8}{\langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2}$$

where a,b are - helicity gravitons

$$\frac{[ab]^8}{[12]^2[23]^2[31]^2}$$

where a,b are + helicity gravitons

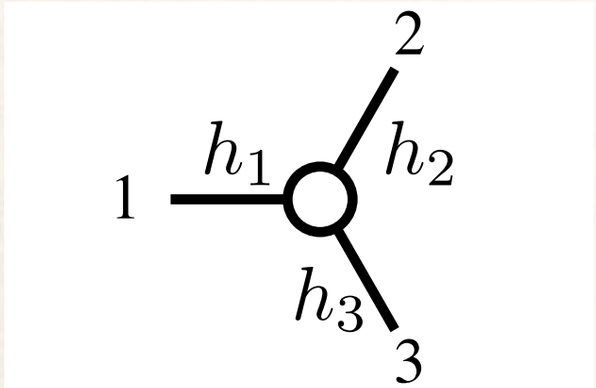
- Note that there is no three-point scattering

  They exist only for complex momenta

- Important input into on-shell methods

General 3pt amplitudes

- Two solutions for 3pt kinematics

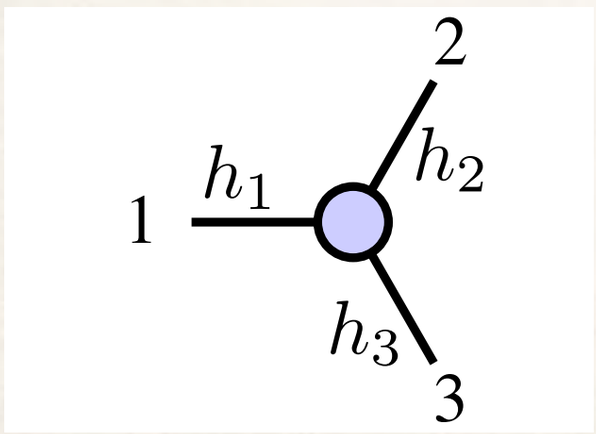


$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$

Under the little group rescaling:

$$A_3(t\lambda_j, t^{-1}\widetilde{\lambda}_j) \sim t^{2h_j} \cdot A_3$$

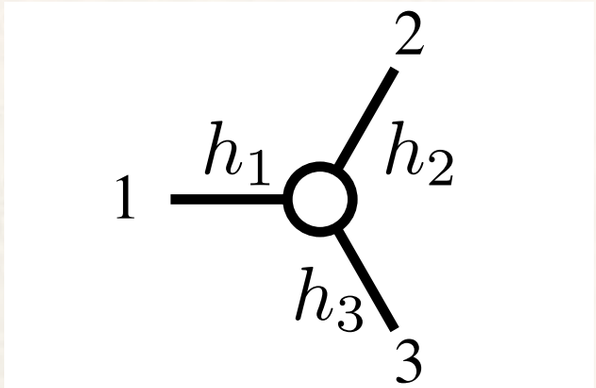
Solve the system of equations



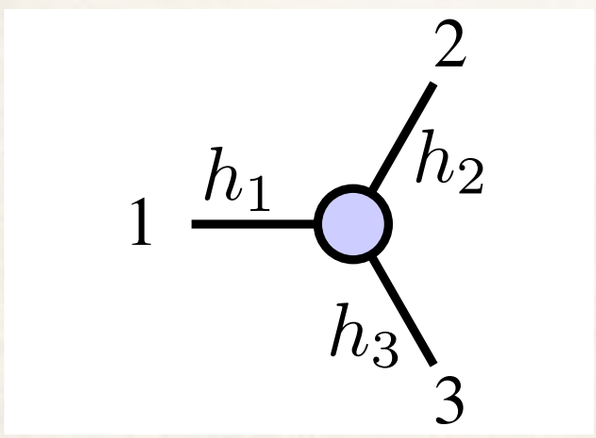
$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

General 3pt amplitudes

- Two solutions for 3pt amplitudes



$$A_3 = [12]^{-h_1-h_2+h_3}[23]^{-h_2-h_3+h_1}[31]^{-h_1-h_3+h_2}$$

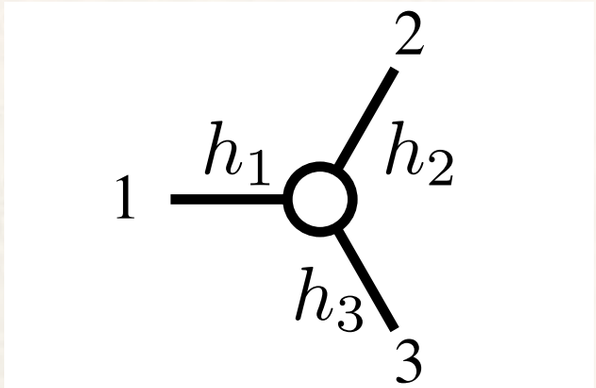


$$A_3 = \langle 12\rangle^{h_1+h_2-h_3}\langle 23\rangle^{h_2+h_3-h_1}\langle 31\rangle^{h_1+h_3-h_2}$$

Which one is correct?

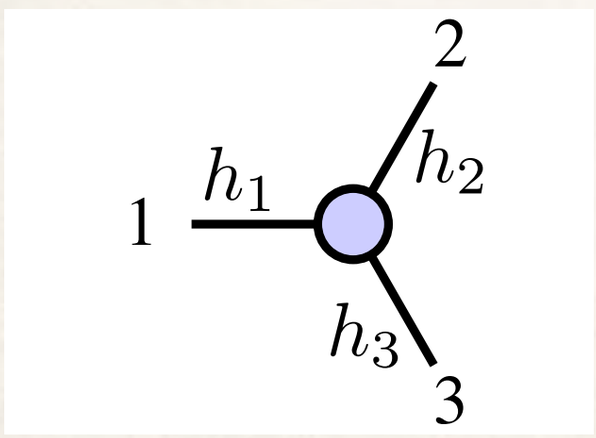
General 3pt amplitudes

- Two solutions for 3pt amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \leq 0$$



$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

$$h_1 + h_2 + h_3 \geq 0$$

Mass dimension must be positive!

All spins allowed

- Note that these formulas are valid for any spins
- For example for amplitude $A_3(1^{0}, 2^{1^+}, 3^{2^+})$

$$A_3 = \frac{\langle 23\rangle \langle 31\rangle^3}{\langle 12\rangle^3}$$

- But we can also do higher spins $A_3(1^{3^+}, 2^{5^+}, 3^{12^-})$

$$A_3 = \frac{\langle 23\rangle^{10} \langle 31\rangle^{14}}{\langle 12\rangle^{20}}$$

- Completely fixed just by kinematics!

Tree-level amplitudes

Feynman diagrams

- Yang-Mills Lagrangian

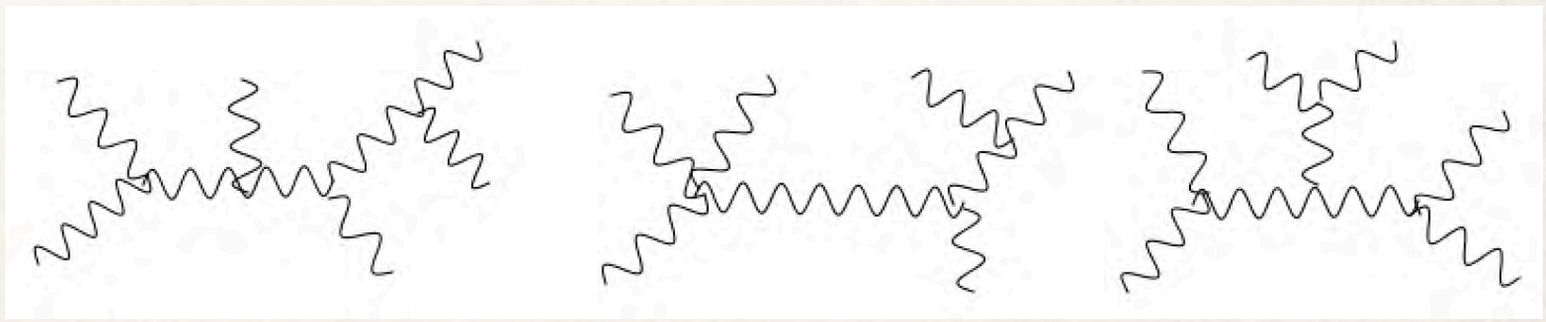




$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sim (\partial A)^2 + A^2 \partial A + A^4$$

$$\sim f^{abc}\, g_{\mu\nu} p_\alpha \qquad\qquad \sim f^{abe} f^{cde}\, g_{\mu\nu} g_{\alpha\beta}$$

- Draw diagrams
  Feynman rules
  Sum everything

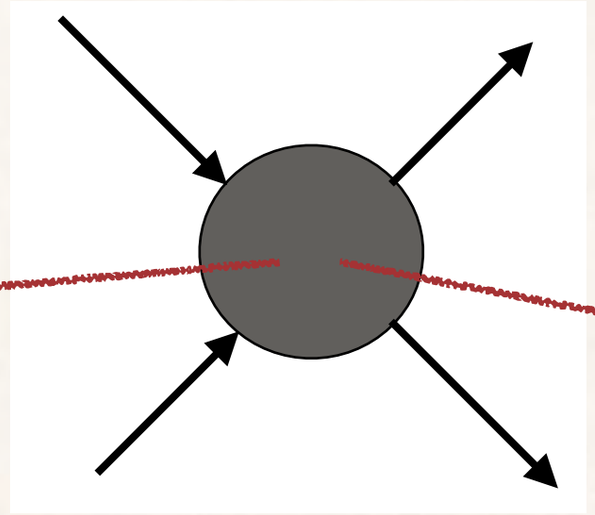


Change of strategy

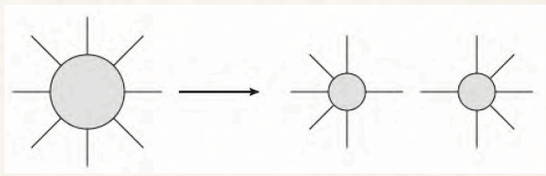
What is the scattering amplitude?

Feynman diagrams





Unique object fixed by physical properties



Was not successful

(1960s)



Modern methods use both:

- Calculate the amplitude directly
- Use perturbation theory

Locality and unitarity

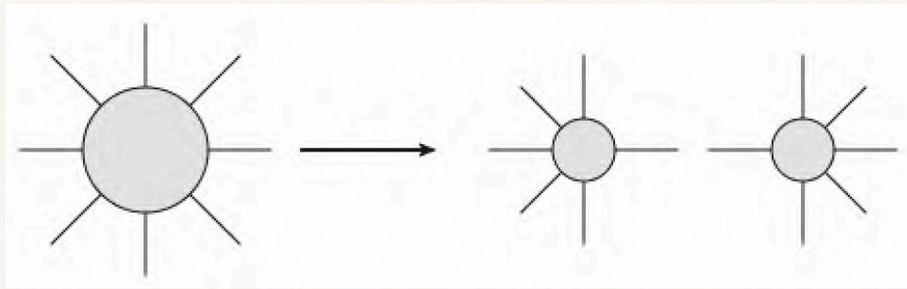
- Only poles: Feynman propagators

Locality

$$\frac{1}{P^2} \quad \text{where} \quad P = \sum_{k \in \mathcal{P}} p_k$$

- On the pole

Unitarity



$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

Feynman diagrams recombine on both sides into amplitudes

Factorization on the pole

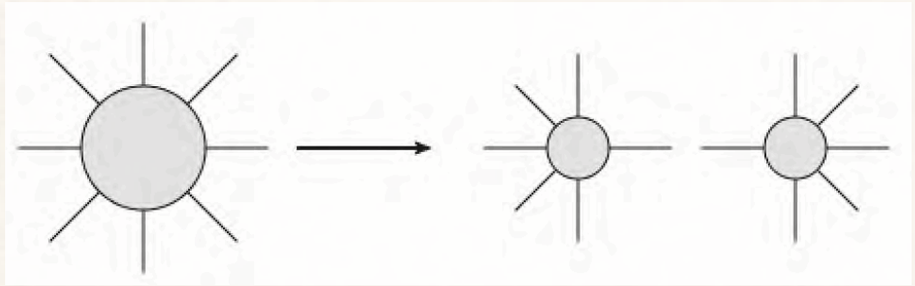
On $P^2 = 0$

$$\text{Res}\,\mathcal{M} = \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

- For $P^2 = 0$ the internal leg: on-shell physical particle
- Both sub-amplitudes are on-shell, gauge invariant
- On-shell data: statement about on-shell quantities

On-shell constructibility

- Factorization of tree-level amplitudes



$$\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$$

- On-shell constructibility: factorizations fix the answer

- Write a proposal tree-level amplitude $\widetilde{\mathcal{M}}$
  - On-shell gauge invariant function, correct weights
  - It factorizes properly on all channels

The amplitude is uniquely specified by these properties

On-shell constructibility

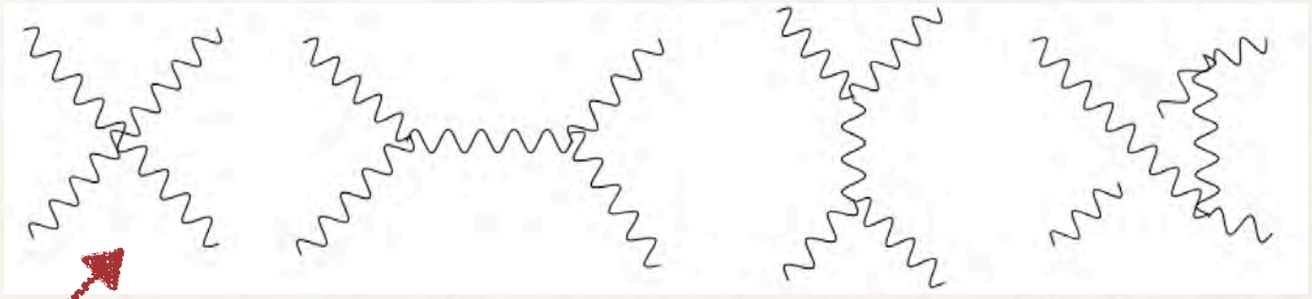
- This is obviously a theory specific statement

- Theories with contact terms might not be constructible

- Naively, this is false for Yang-Mills theory

$$gg \to gg$$

Four point amplitude



Contact term

On-shell constructibility

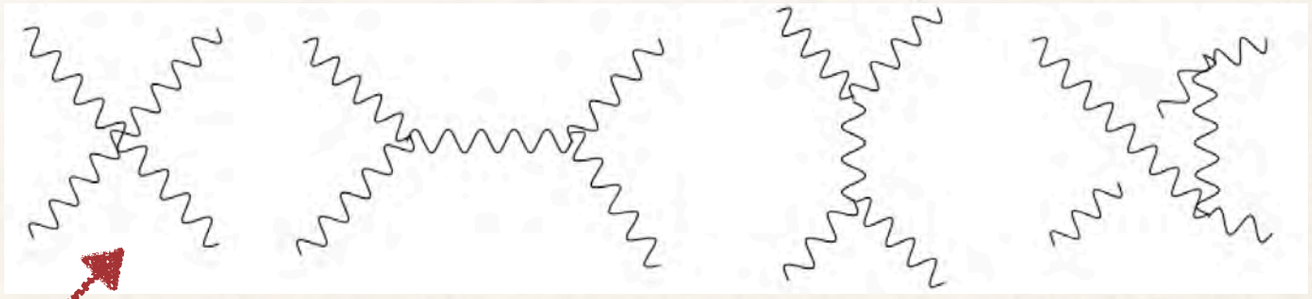
- This is obviously a theory specific statement

- Theories with contact terms might not be constructible

- Naively, this is false for Yang-Mills theory

$$gg \to gg$$

Four point amplitude



Contact term

Imposing gauge invariance fixes it

On-shell constructibility

- In gravity we have infinity tower of terms

$$\mathcal{L} \sim \sqrt{g}\, R \sim h^2 + h^3 + h^4 + \dots$$

- Only h^3 terms important, others fixed by diffeomorphism symmetry

- On-shell constructibility of Yang-Mills, GR, SM

Only function which factorizes properly on all poles is the amplitude.

Four point test

From 3pt to 4pt

- Three point amplitudes exist for all spins

- For 4pt amplitude: we have a powerful constraint

$$A_4 \xrightarrow[s=0]{} A_3 \frac{1}{s} A_3$$

This must be true on all channels

- This will immediately kill most of the possibilities

- We are left with spectrum of spins: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$

Three point of spin S

- I will discuss amplitudes of single spin S particle
- For 3pt amplitudes we get

$$A_3 = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^S \qquad A_3 = \left(\frac{[12]^3}{[23][31]}\right)^S$$

minimal

powercounting

$$(--+) \qquad (++-)$$

- There exist also non-minimal amplitudes

$$A_3 = (\langle 12\rangle\langle 23\rangle\langle 31\rangle)^S \qquad A_3 = ([12][23][31])^S$$

$$(---) \qquad (+++)$$

Four point amplitude

- Let us consider a 4pt amplitude of particular helicities

$$A_4(--++)$$

- Mandelstam variables:

$$s = (p_1 + p_2)^2 = \langle 12\rangle[12] = \langle 34\rangle[34]$$
$$t = (p_1 + p_4)^2 = \langle 14\rangle[14] = \langle 23\rangle[23]$$
$$u = (p_1 + p_3)^2 = \langle 13\rangle[13] = \langle 24\rangle[24]$$

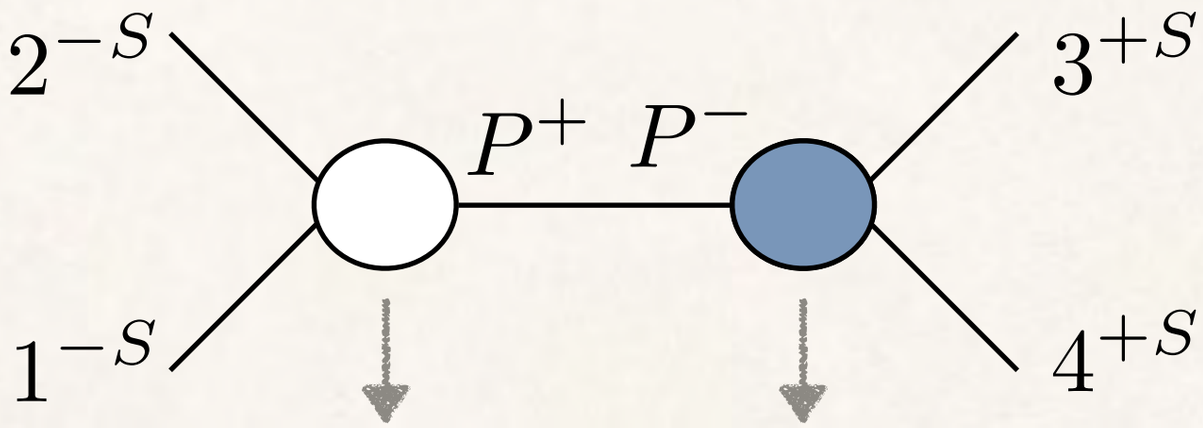
- One can show that the little group dictates:

$$A_4 = (\langle 12\rangle[34])^{2S} \cdot F(s,t)$$

- It must be consistent with factorizations

s-channel constraint

- The s-channel factorization dictates $P = 1 + 2 = -3 - 4$

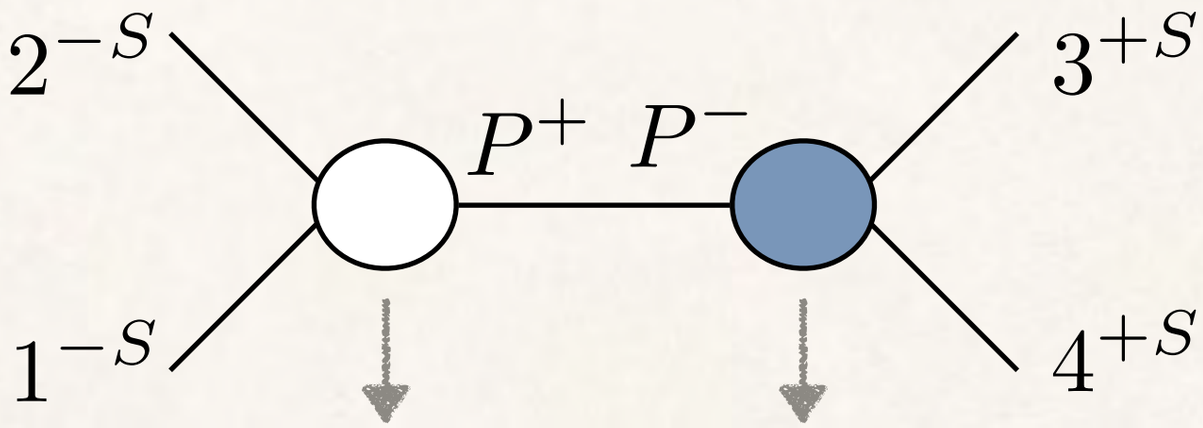


$$A_4 \to \left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s} \left(\frac{[34]^3}{[3P][4P]}\right)^S \quad \text{on s=0}$$

Note: $s = \langle 12\rangle[12] = \langle 34\rangle[34]$

s-channel constraint

- The s-channel factorization dictates $P = 1 + 2 = -3 - 4$



$$A_4 \to \left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s} \left(\frac{[34]^3}{[3P][4P]}\right)^S \quad \text{on s=0}$$

Note: $s = \langle 12\rangle[12] = \langle 34\rangle[34]$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^3}{\langle 1P\rangle[3P]\langle 2P\rangle[4P]}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^3}{\langle 12\rangle[23]\langle 23\rangle[34]}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^2}{\langle 23\rangle[23]}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get $$\frac{1}{s}\left(\frac{(\langle 12\rangle[34])^2}{t}\right)^S$$

s-channel constraint

$$\left(\frac{\langle 12\rangle^3}{\langle 1P\rangle\langle 2P\rangle}\right)^S \frac{1}{s}\left(\frac{[34]^3}{[3P][4P]}\right)^S$$

- Rewrite using momentum conservation:

$$\langle 1P\rangle[3P] = -\langle 1|P|3] = -\langle 1|1+2|3] = \langle 12\rangle[23]$$
$$\langle 2P\rangle[4P] = -\langle 2|P|4] = \langle 2|3+4|4] = \langle 23\rangle[34]$$

- We get $\left(\langle 12\rangle[34]\right)^{2S} \cdot \frac{1}{s\,t^S}$

Note: $t = -u$

"Trivial" helicity factor

Important piece

Comparing channels

- On s-channel we got: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\dfrac{1}{s\,t^S}}$
- On t-channel we would get: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\dfrac{1}{t\,s^S}}$
- Require simple poles: $\dfrac{1}{s}, \dfrac{1}{t}, \dfrac{1}{u}$ and search for $F(s,t,u)$

Comparing channels

- On s-channel we got: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\dfrac{1}{s\,t^S}}$
- On t-channel we would get: $A_4 \to (\langle 12\rangle[34])^{2S} \cdot \boxed{\dfrac{1}{t\,s^S}}$
- Require simple poles: $\dfrac{1}{s}, \dfrac{1}{t}, \dfrac{1}{u}$ and search for $F(s,t,u)$
- There are only two solutions:

$$F(s,t,u) = \frac{1}{s} + \frac{1}{t} + \frac{1}{u} \qquad F(s,t,u) = \frac{1}{stu}$$

spin 0 (ϕ^3) spin 2 (GR)

Where are gluons (spin-1)?

- Need to consider multiplet of particles

$$A_3 = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^S f^{a_1a_2a_3} \qquad A_3 = \left(\frac{[12]^3}{[23][31]}\right)^S f^{a_1a_2a_3}$$

- The same check gives us S=1 and requires

$$f^{a_1a_2a_P} f^{a_3a_4a_P} + f^{a_1a_4a_P} f^{a_2a_3a_P} = f^{a_1a_3a_P} f^{a_2a_4a_P}$$

and the result corresponds to
SU(N) Yang-Mills theory

Power of 4pt check

- We can apply this check for cases with mixed particle content:
  - Spin >2 still not allowed
  - Spin 2 is special: only one particle and it couples universally to all other particles
  - We get various other constraints on interactions (of course all consistent with known theories)
- General principles very powerful

Recursion relations

Tree level amplitudes

- Tree-level amplitude is a rational function of kinematics

$$A = \sum(\text{Feyn. diag}) = \frac{N}{\prod_j P_j^2}$$

momenta
polarization vectors

Feynman propagators

$$P_j = \sum_k p_k$$

- Only poles, no branch cuts

- Gauge invariant object: use spinor helicity variables

Reconstruction of the amplitude

- Amplitude on-shell constructible: fixed only from factorizations: try to reconstruct it

  "Integrate the relation" $\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

- First guess: $\mathcal{M} = \sum_P \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

Reconstruction of the amplitude

- Amplitude on-shell constructible: fixed only from factorizations: try to reconstruct it

"Integrate the relation" $\mathcal{M} \xrightarrow[P^2=0]{} \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$

- First guess: $\mathcal{M} = \sum_P \mathcal{M}_L \frac{1}{P^2} \mathcal{M}_R$ **WRONG**

Overlapping factorization channels

- Solution: shift external momenta

Momentum shift

- Let us shift two external momenta

$$\lambda_1 \to \lambda_1 - z\lambda_2 \qquad \widetilde{\lambda}_1 \to \widetilde{\lambda}_1$$
$$\lambda_2 \to \lambda_2 \qquad \widetilde{\lambda}_2 \to \widetilde{\lambda}_2 + z\widetilde{\lambda}_1$$

- Momentum is conserved, stays on-shell

$$(\lambda_1 - z\lambda_2)\widetilde{\lambda}_1 + \lambda_2(\widetilde{\lambda}_2 + z\widetilde{\lambda}_1) = \lambda_1\widetilde{\lambda}_1 + \lambda_2\widetilde{\lambda}_2$$

- This corresponds to shifting

$$p_1, p_2, \epsilon_1, \epsilon_2$$

Shifted amplitude

- On-shell tree-level amplitude with shifted kinematics

$$A_n(z) = A(\hat{p}_1(z), \hat{p}_2(z), p_3, \ldots, p_n)$$

- Analytic structure

$$A_n(z) = \frac{N(z)}{\prod_j P_j(z)^2}$$

- Location of poles:

$$P_j(z) = P_j - z\lambda_2\widetilde{\lambda}_1 \quad \text{if} \quad p_1 \in P_j$$

$$P_j(z) = P_j + z\lambda_2\widetilde{\lambda}_1 \quad \text{if} \quad p_2 \in P_j$$

$$P_j(z) = P_j \qquad \text{otherwise}$$

Shifted amplitude

- On the pole if $p_1 \in P_j$

$$P_j(z)^2 = P_j^2 - 2z\langle 1|P_j|2] = 0$$

$$z = \frac{P_j^2}{2\langle 1|P_j|2]} \equiv z_j$$

- Shifted amplitude:

$$A_n(z) = \frac{N(z)}{\prod_j P_j(z)^2}$$

location of poles

Residue theorem

- Shifted amplitude $A_n(z) = \dfrac{N(z)}{\prod_k (z - z_k)}$
- Let us consider the contour integral

$$\int \frac{dz}{z} A_n(z) = 0 \qquad \text{No pole at } z \to \infty$$

- Original amplitude $A_n = A_n(z = 0)$ ← Residue at $z = 0$
- Residue theorem: $A_n + \sum_k \mathrm{Res}\left(\dfrac{A_n(z)}{z}\right)\Big|_{z=z_k} = 0$

Residue theorem

$$A_n = -\sum_k \mathrm{Res}\left(\frac{A_n(z)}{z}\right)\Big|_{z=z_k}$$

Residue on the pole $\quad P_j(z)^2 = 0$

- Unitarity of shifted tree-level amplitude

$$A_n(z) \xrightarrow[P_j(z)^2=0]{} A_L(z)\frac{1}{P_j(z)^2}A_R(z)$$

Residue theorem

$$A_n = -\sum_k \mathrm{Res}\left(\frac{A_n(z)}{z}\right)\Big|_{z=z_k}$$

Residue on the pole $P_j(z)^2 = 2\langle 1|P_j|2](z_j - z) = 0$

- Unitarity of shifted tree-level amplitude $z_j = \dfrac{P_j^2}{2\langle 1|P_j|2]}$

$$A_n(z) \xrightarrow[z=z_j]{} A_L(z_j)\frac{1}{2\langle 1|P_j|2]}A_R(z_j)$$

Residue theorem

$$A_n = -\sum_k \text{Res}\left(\frac{A_n(z)}{z}\right)\Big|_{z=z_k}$$

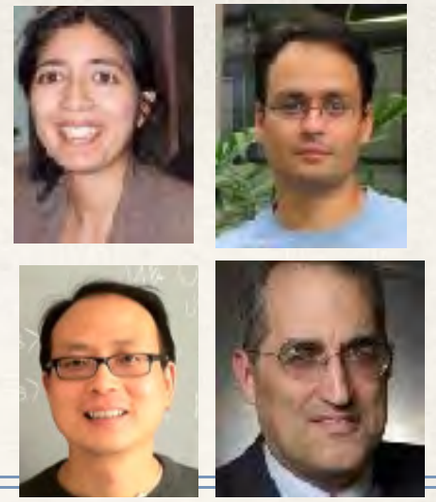
$$A_L(z_j)\frac{1}{2\langle 1|P_j|2]}A_R(z_j) \times \frac{2\langle 1|P_j|2]}{P_j^2} = A_L(z_j)\frac{1}{P_j^2}A_R(z_j)$$

Final formula

$$A_n = -\sum_j A_L(z_j)\frac{1}{P_j^2}A_R(z_j) \qquad z_j = \frac{P_j^2}{2\langle 1|P_j|2]}$$

BCFW recursion relations

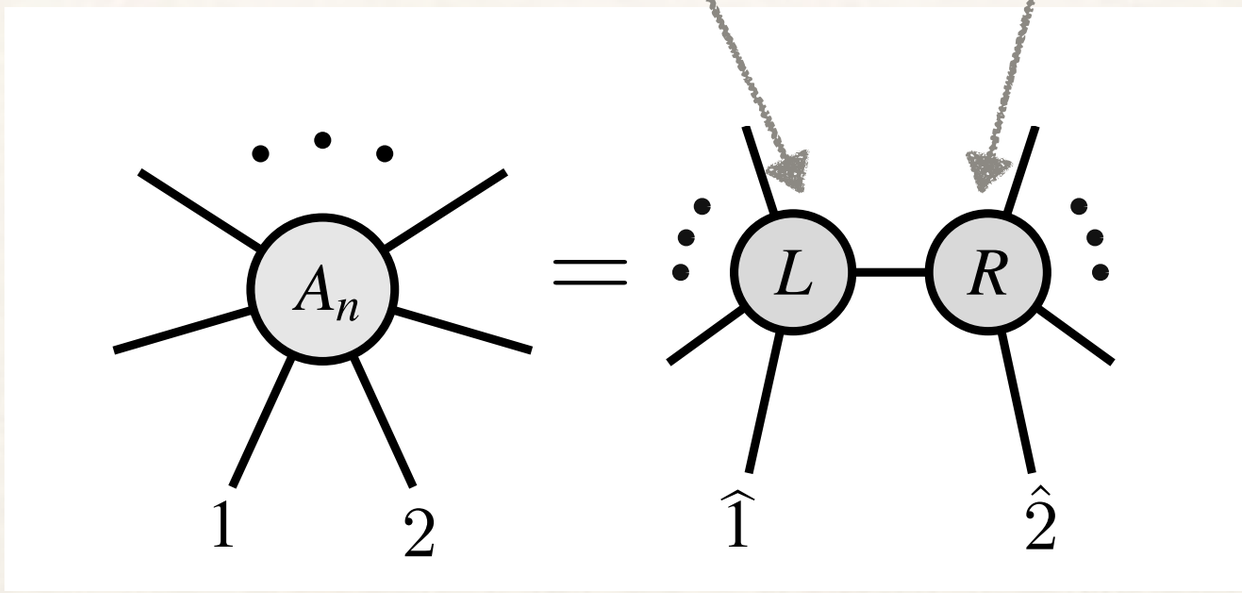
(Britto, Cachazo, Feng, Witten, 2005)



$$A_n = -\sum_j A_L(z_j)\frac{1}{P_j^2}A_R(z_j)$$

$$z_j = \frac{P_j^2}{2\langle 1|P_j|2]}$$

Chosen such that internal line is on-shell



Sum over all distributions of legs keeping 1,2 on different sides

Comment on applicability

- The crucial property is $A_n(z) \to 0 \quad \text{for} \quad z \to \infty$

- In Yang-Mills theory this is satisfied if

$$\lambda_1 \to \lambda_1 - z\lambda_2 \quad \longleftarrow \quad \text{Helicity +}$$
$$\widetilde{\lambda}_2 \to \widetilde{\lambda}_2 + z\widetilde{\lambda}_1 \quad \longleftarrow \quad \text{Helicity -}$$

- Same is true for Einstein gravity, and many others

- This means that amplitudes in these theories are fully specified by residues on their poles

Generalizations

- In Standard Model and other theories more general recursion relations needed: shift more momenta

- Include masses: go back to momenta

$$p_1 \to p_1 + zq \qquad q^2 = (p_1 \cdot q) = (p_2 \cdot q) = 0$$
$$p_2 \to p_2 - zq$$

Shifted momenta on-shell, q completely fixed

- Extension to effective field theories

Unitarity methods

One-loop amplitudes

- Sum of Feynman diagrams

$$\mathcal{M}^{1-loop} = \sum_j \int d\mathcal{I}_j \quad \text{where} \quad d\mathcal{I}_j = d^4\ell\, \mathcal{I}_j$$

- Re-express as basis of canonical integrals

$$\mathcal{M}^{1-loop} = \sum_j a_j \int d\mathcal{I}_j^{(4)} + \sum_j b_j \int d\mathcal{I}_j^{(3)} + \sum_j c_j \int d\mathcal{I}_j^{(2)} + \mathcal{R}$$

Rational

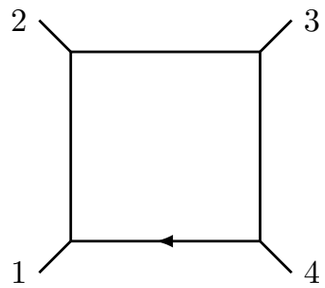
Box

Triangle

Bubble

One loop amplitudes

- Box integral



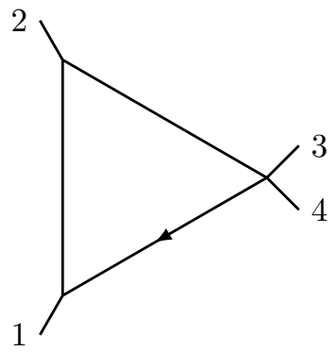
$$I = \frac{d^4\ell \; st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

Tadpoles and other integrals

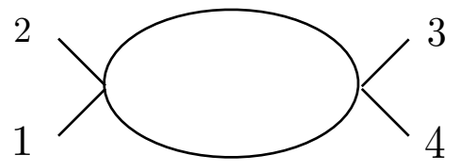
↓

Vanish in dim reg

- Triangle and box integrals



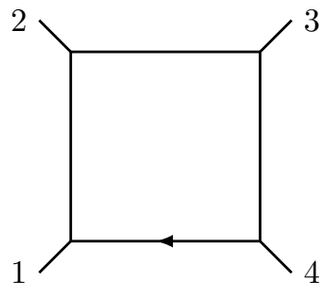
$$I = \frac{d^4\ell \; s}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2}$$



$$I = \frac{d^4\ell \, s}{\ell^2(\ell+k_1+k_2)^2}$$

One loop amplitudes

- Box integral



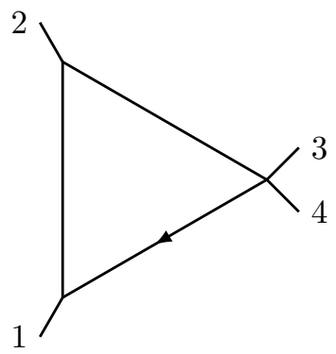
$$I = \frac{d^4\ell \; st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

Tadpoles and other integrals

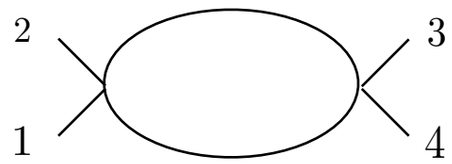
↓

Vanish in dim reg

- Triangle and box integrals



$$I = \frac{d^4\ell \; s}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2}$$



$$I = \frac{d^4\ell \, s}{\ell^2(\ell+k_1+k_2)^2}$$

UV divergent

(Super) Yang Mills amplitudes

- One-loop expansion in SYM theory

$$\mathcal{M} = \sum \text{Boxes} + \sum \text{Triangle} + \sum \text{Bubble} + \text{Rational}$$

Pure Yang-Mills (massless QCD)

(Super) Yang Mills amplitudes

- One-loop expansion in SYM theory

$$\mathcal{M} = \sum \text{Boxes} + \sum \text{Triangle} + \sum \text{Bubble} + \text{Rational}$$

N=1 and N=2 Super Yang-Mills

(Super) Yang Mills amplitudes

- One-loop expansion in SYM theory

$$\mathcal{M} = \sum \text{Boxes} + \sum \text{Triangle} + \sum \text{Bubble} + \text{Rational}$$

N=4 Super Yang-Mills

- Note that it is UV finite at 1-loop, but also all loops

One loop expansion

- One-loop expansion

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

One loop expansion

- One-loop expansion

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

How to calculate these coefficients?

How to calculate this function?

One loop expansion

- One-loop expansion

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

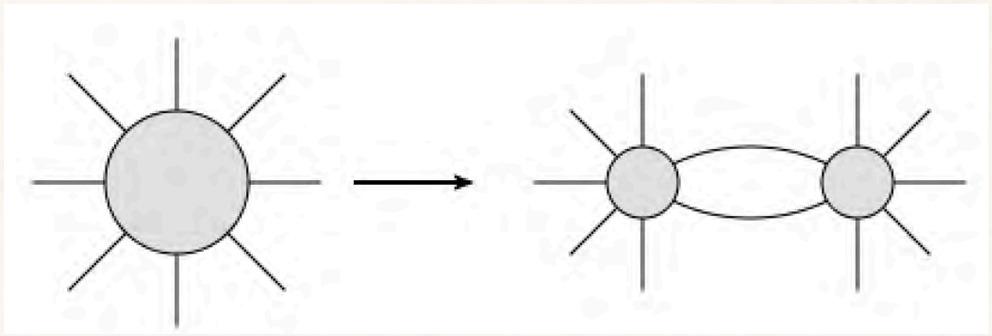
How to calculate these coefficients?

How to calculate this function?

Unitarity methods

One loop unitarity

- Analogue of tree-level unitarity at one-loop



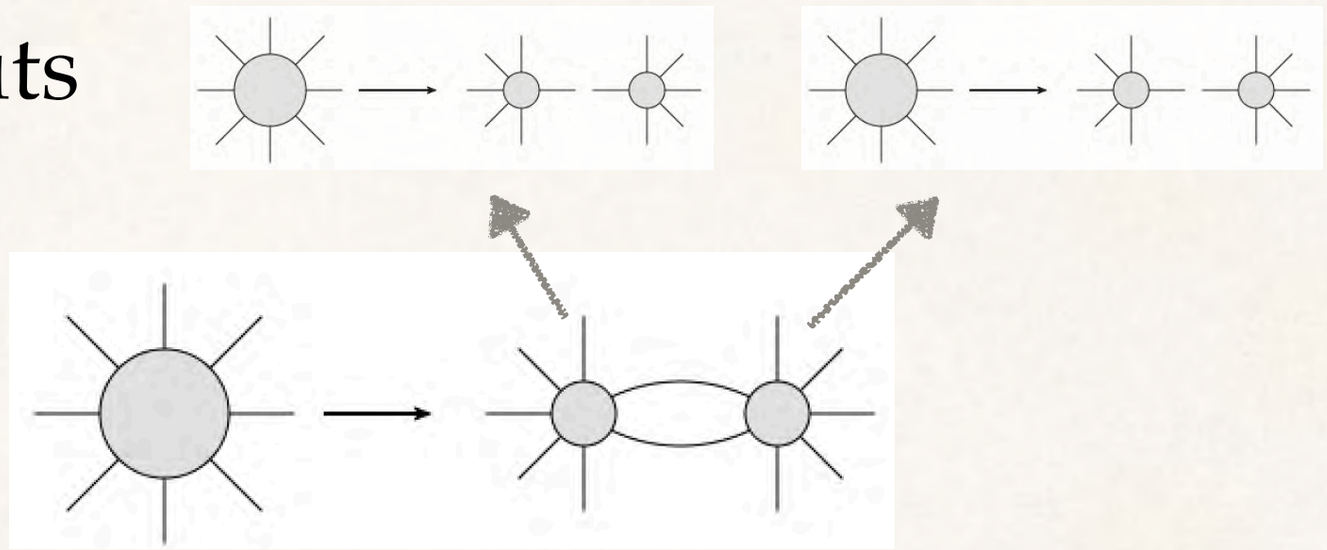
$$\mathcal{M}^{1-loop} \xrightarrow[\ell^2=(\ell+Q)^2=0]{} \mathcal{M}_L^{tree} \frac{1}{\ell^2(\ell+Q)^2} \mathcal{M}_R^{tree}$$

Unitarity cut

- In general $\quad \text{Cut} \leftrightarrow \ell^2 = 0$

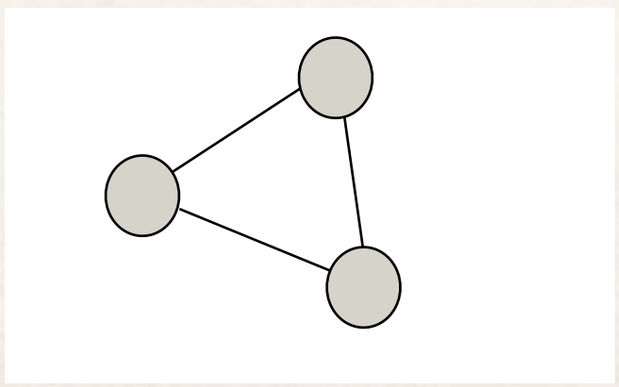
One-loop unitarity

- Higher cuts



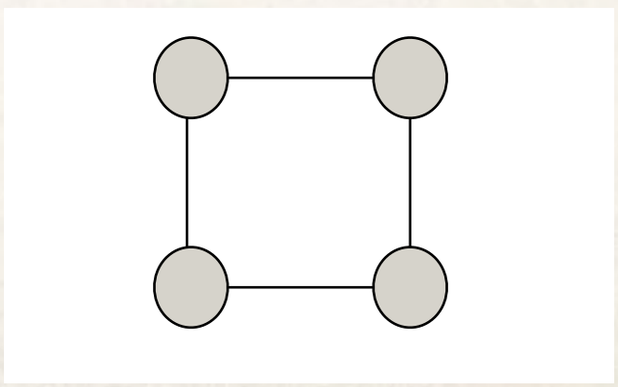
Triple cut

$$\ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = 0$$



Quadruple cut

$$\ell^2 = (\ell + Q_1)^2 = (\ell + Q_2)^2 = (\ell + Q_3)^2 = 0$$



Fixing coefficients

- Perform cut on both side of equation

$$\mathcal{M} = \sum_j a_j \text{Boxes}_j + \sum_j b_j \text{Triangle}_j + \sum_j c_j \text{Bubble}_j + \text{Rational}$$

Product of trees Linear combination of coefficients

- Example: Quadruple cut - only one box contributes

$$\mathcal{M}_1^{tree} \mathcal{M}_2^{tree} \mathcal{M}_3^{tree} \mathcal{M}_4^{tree} = a_j$$

- All coefficients a_j, b_j, c_j can be obtained

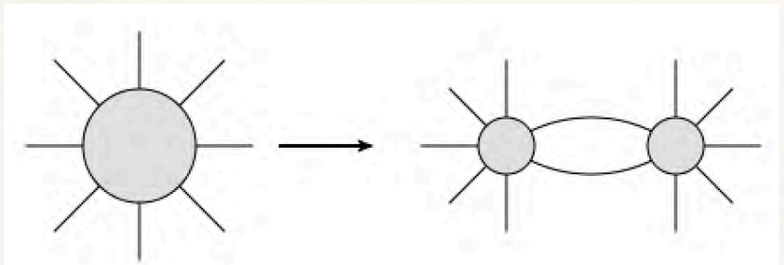
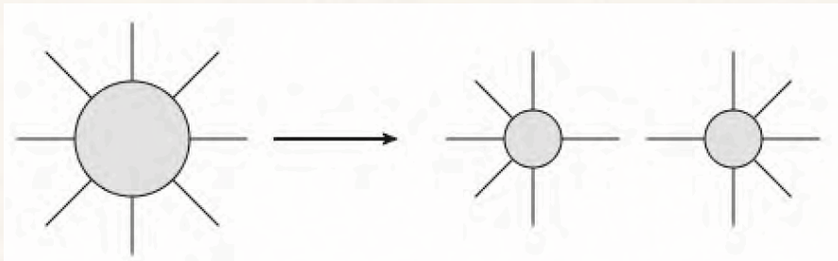
Unitarity methods

(Bern, Dixon, Kosower)

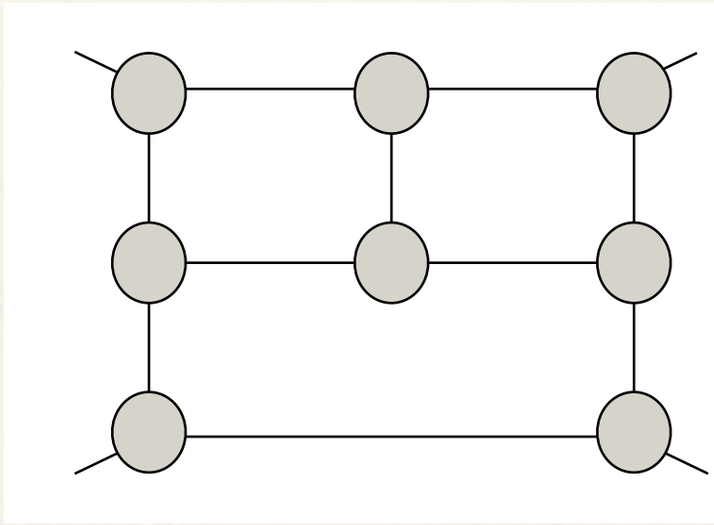




- We can iterate both types of cuts

- Stop when all propagators are cut: maximal cut



Product of 3pt on-shell amplitudes

Unitarity methods

(Bern, Dixon, Kosower)





- Expansion of the amplitude

$$\mathcal{M}^{\ell-loop} = \sum_j a_j \int d\mathcal{I}_j$$

Cuts give product of trees

Linear combinations of coefficients a_j

- Very successful method for loop amplitudes in different theories

- Practical problems:
  - Find basis of integrals
  - Solve (long) system of equations

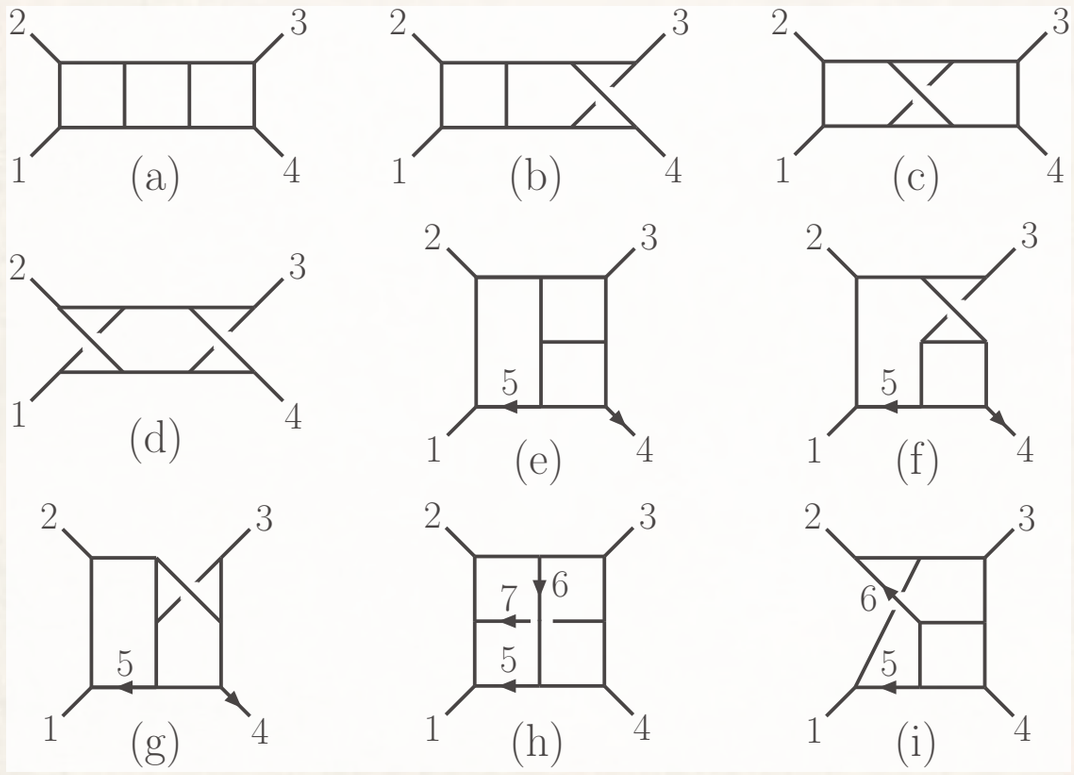
Unitarity methods

(Bern, Dixon, Kosower)

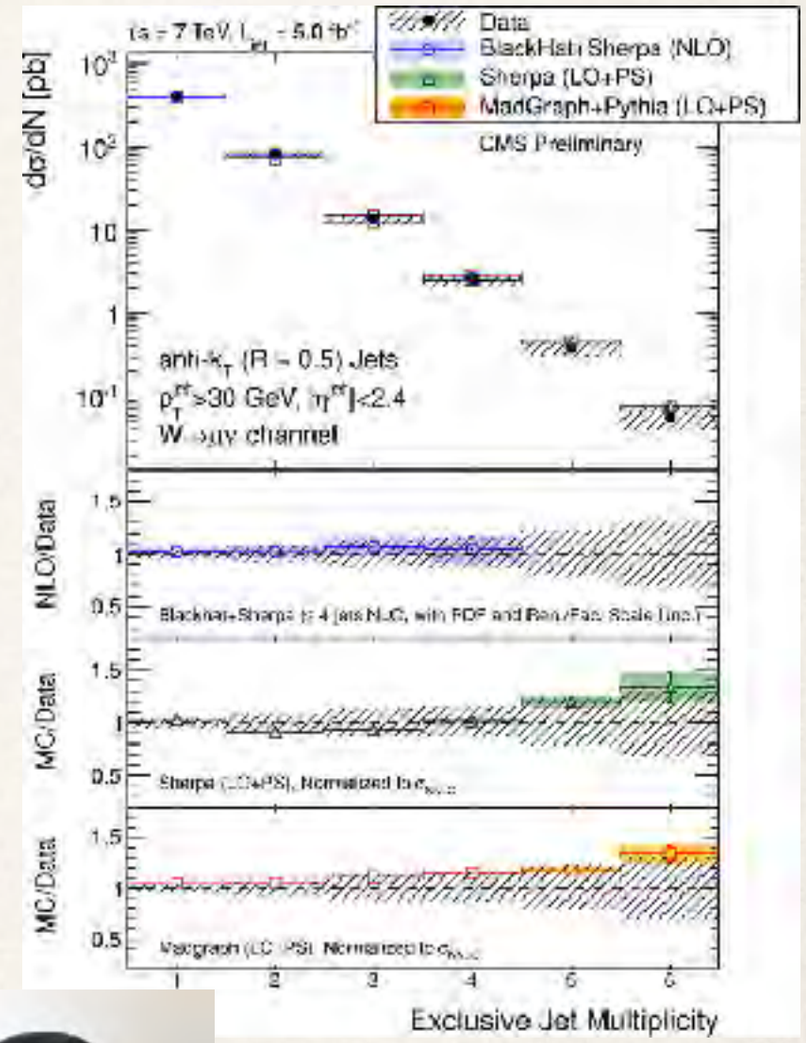


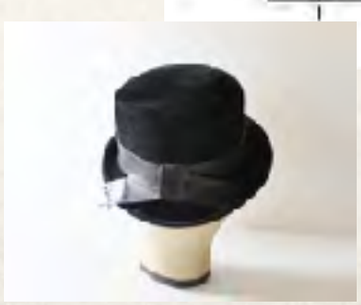


- Results in SUSY theories and QCD



Basis of integrals for 3-loop amplitudes in N=4 SYM and N=8 SUGRA





Black Hat

On-shell good, off-shell bad

- Feynman diagrams: off-shell objects

- Unitarity methods: $\mathrm{Cut}[\mathcal{M}] = \mathrm{Cut}[\mathrm{Basis\ of\ integrals}]$

Off-shell objects

On-shell objects

Locality
Unitarity

- Recursion relations

$$\mathcal{M} \sim \mathcal{M}_L \, \mathcal{M}_R$$

On-shell objects

Locality lost
Unitarity

- Next direction: loosing manifest locality and unitarity

Thank you for attention!



Grassmannian Geometry of Scattering Amplitudes

LECTURE 2

Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP), UC Davis

Amplitudes School, Edinburgh, July 2017

Atoms of amplitudes

What are natural gauge invariant objects?

Atoms of amplitudes

What are natural gauge invariant objects?

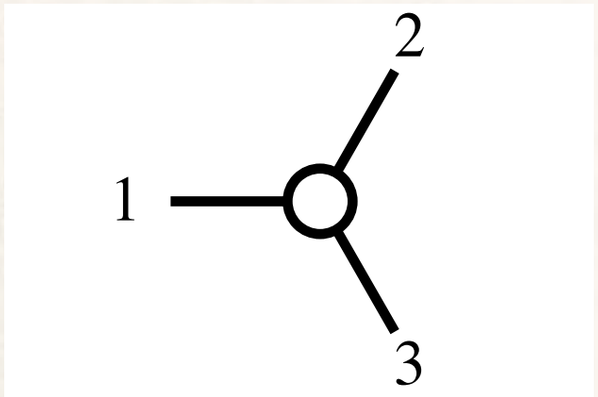
Scattering amplitudes

- Recursion relations, unitarity methods: products of amplitudes
- Iterative procedure: reduces to elementary amplitudes
- In most interesting theories these are three point

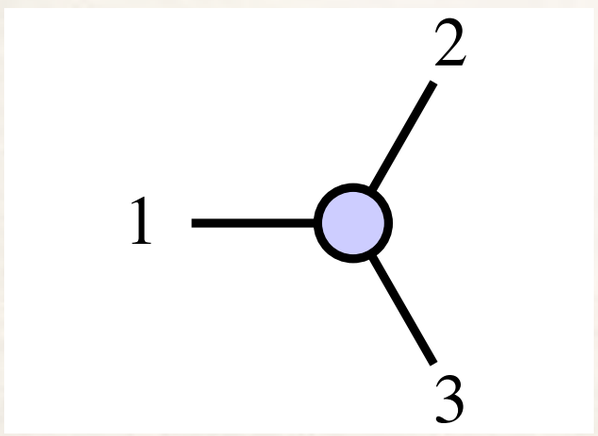
On-shell diagrams

Three point kinematics

- Two options



$$\lambda_1 \sim \lambda_2 \sim \lambda_3$$



$$\widetilde{\lambda}_1 \sim \widetilde{\lambda}_2 \sim \widetilde{\lambda}_3$$

Spinor helicity variables

$$p^\mu = \sigma^\mu_{a\dot{a}} \lambda_a \widetilde{\lambda}_{\dot{a}}$$

$$\langle 12 \rangle = \epsilon_{ab} \lambda_{1a} \lambda_{2b}$$

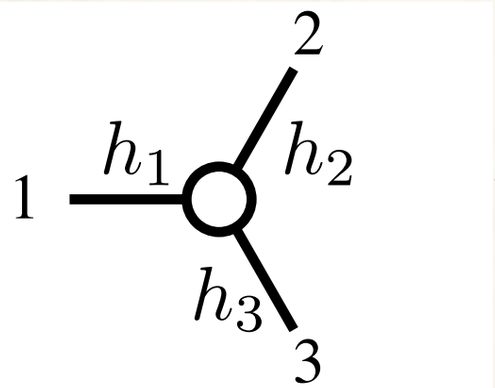
$$[12] = \epsilon_{\dot{a}\dot{b}} \lambda_{1\dot{a}} \lambda_{2\dot{b}}$$

Two solutions for 3pt kinematics

$$p_1^2 = p_2^2 = p_3^2 = (p_1 + p_2 + p_3) = 0$$

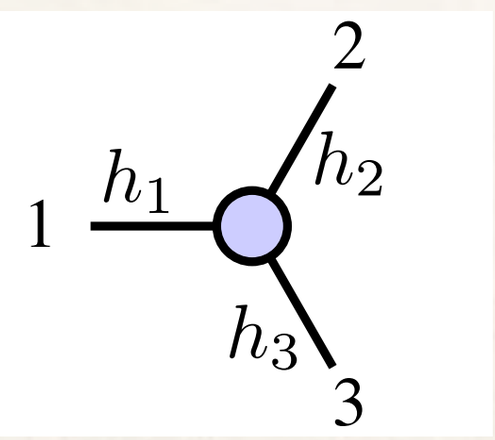
Three point amplitudes

- Two solutions for amplitudes



$$A_3 = [12]^{+h_1+h_2-h_3}[23]^{-h_1+h_2+h_3}[31]^{+h_1-h_2+h_3}$$

$$h_1 + h_2 + h_3 \geq 0$$



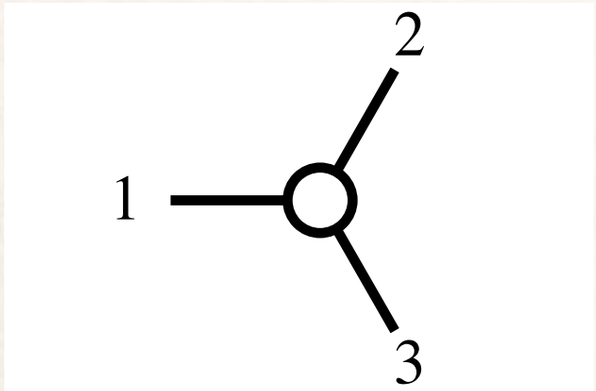
$$A_3 = \langle 12\rangle^{-h_1-h_2+h_3}\langle 23\rangle^{+h_1-h_2-h_3}\langle 31\rangle^{-h_1+h_2-h_3}$$

$$h_1 + h_2 + h_3 \leq 0$$

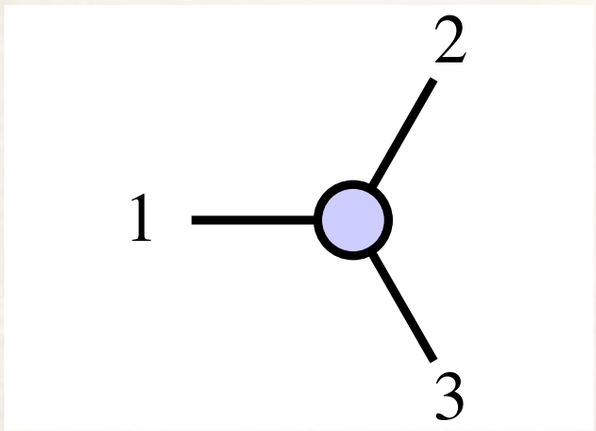
Supersymmetry: amplitudes of super-fields
(all component fields included)

Three point amplitudes

- In N=4 SYM: no need to specify helicities



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$



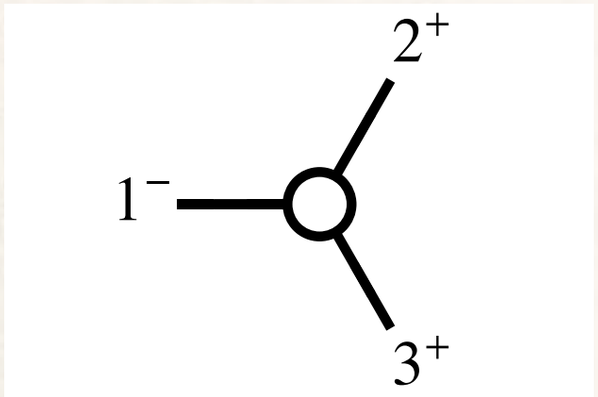
$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Easy book-keeping

Fully fixed in any QFT up to coupling

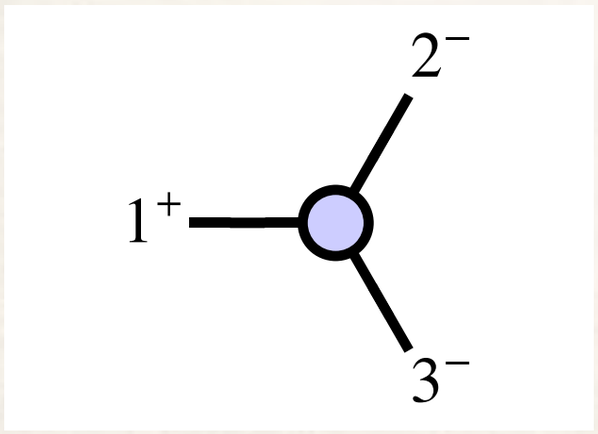
Three point amplitudes

- Two super amplitudes



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$

$\widetilde{\eta}_1^4$ component $\longrightarrow$ $[23]^4$

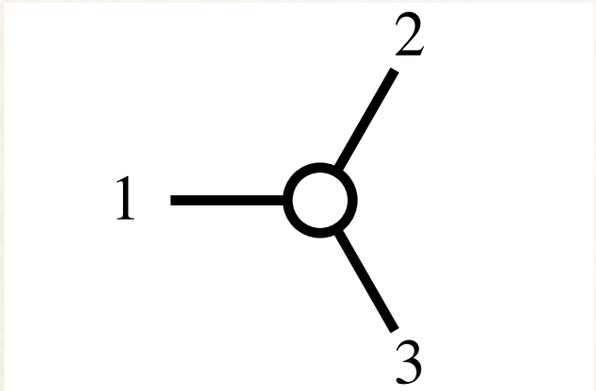


$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

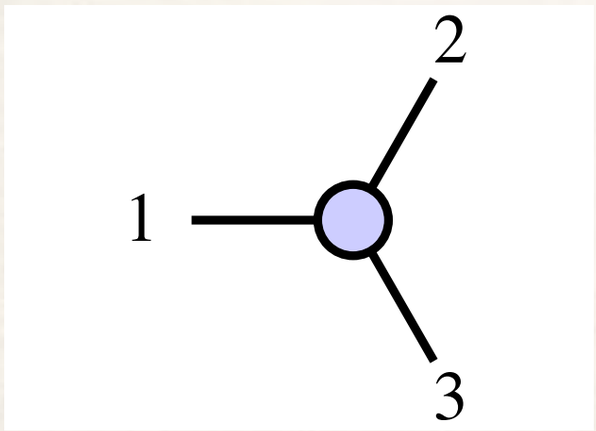
$\widetilde{\eta}_2^4\widetilde{\eta}_3^4$ component $\longrightarrow$ $\langle 23\rangle^4$

Three point amplitudes

- In N=4 SYM: no need to specify helicities



$$\mathcal{A}_3^{(1)} = \frac{\delta^4(p_1+p_2+p_3)\delta^4([23]\widetilde{\eta}_1+[31]\widetilde{\eta}_2+[12]\widetilde{\eta}_3)}{[12][23][31]}$$



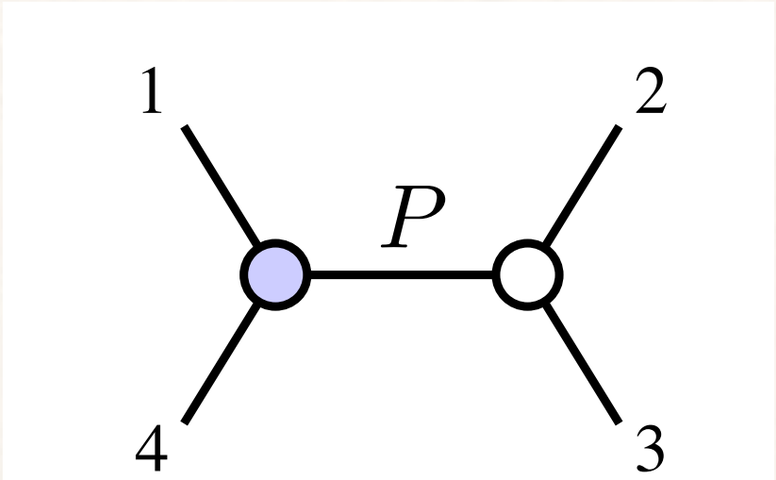
$$\mathcal{A}_3^{(2)} = \frac{\delta^4(p_1+p_2+p_3)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Easy book-keeping

Fully fixed in any QFT up to coupling

Gluing three point amplitudes

- Let us build a diagram



Multiply two three point amplitudes

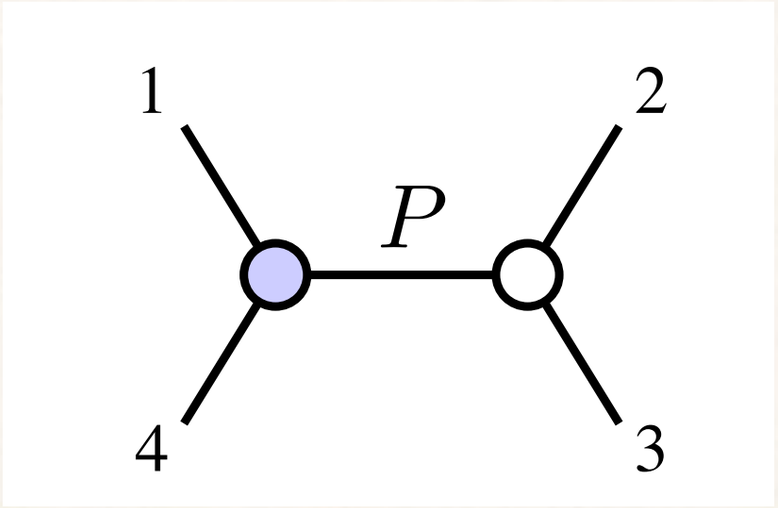
$$= \mathcal{A}_3^{(2)}(14P) \times \mathcal{A}_3^{(1)}(P23)$$

$$= \frac{\delta^4(p_1+p_4+P)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_4\widetilde{\eta}_4+\lambda_P\widetilde{\eta}_P)}{\langle 14\rangle\langle 4P\rangle\langle P1\rangle} \times \frac{\delta^4(p_2+p_3-P)\delta^4(\widetilde{\eta}_P[23]+\widetilde{\eta}_2[3P]+\widetilde{\eta}_3[P2]}{[23][3P][P2]}$$

also $\lambda_P \sim \lambda_2 \sim \lambda_3$ and $\widetilde{\lambda}_1 \sim \widetilde{\lambda}_4 \sim \widetilde{\lambda}_P$

Gluing three point amplitudes

- Let us build a diagram



Multiply two three point amplitudes

$$= \mathcal{A}_3^{(2)}(14P) \times \mathcal{A}_3^{(1)}(P23)$$

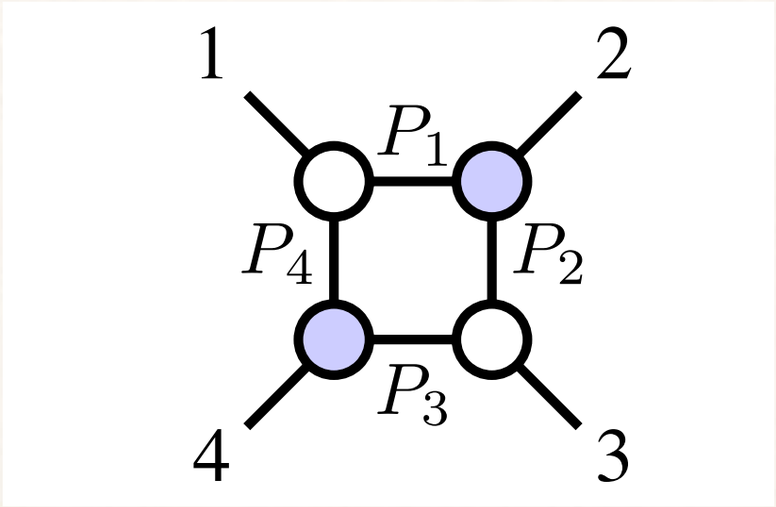
$$= \frac{\delta^4(p_1+p_2+p_3+p_4)\delta^8(\lambda_1\widetilde{\eta}_1+\lambda_2\widetilde{\eta}_2+\lambda_3\widetilde{\eta}_3+\lambda_4\widetilde{\eta}_4)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \delta((p_2+p_3)^2)$$

$$= \mathcal{A}_4^{(2)}(1234) \times \delta((p_2+p_3)^2)$$

Four point tree level amplitude on factorization channel

Gluing three point amplitudes

- Let us build a diagram

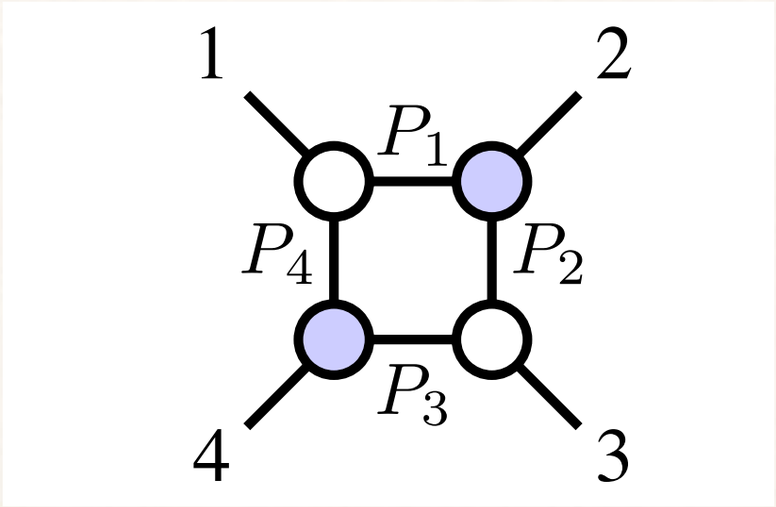


Multiply four three point amplitudes

$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

Gluing three point amplitudes

- Let us build a diagram



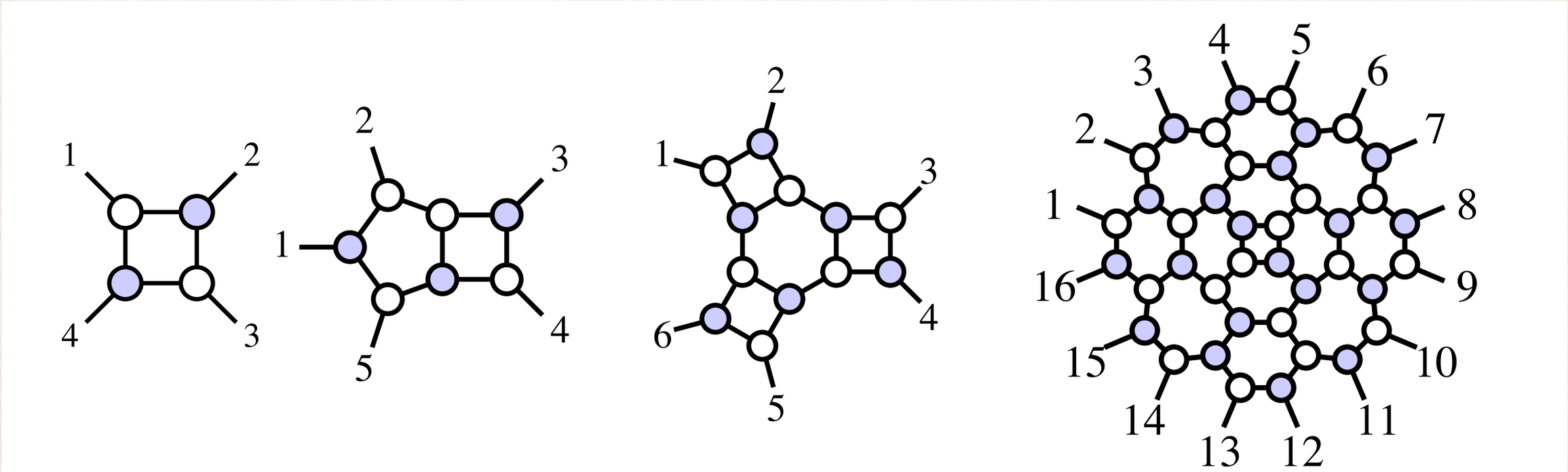
Multiply four three point amplitudes

$$= \mathcal{A}_3^{(1)}(1P_1P_4) \times \mathcal{A}_3^{(2)}(2P_2P_1) \times \mathcal{A}_3^{(1)}(3P_3P_2) \times \mathcal{A}_3^{(2)}(4P_4P_3)$$

$$= \mathcal{A}_4(1234)$$

On-shell diagrams

- Draw arbitrary graph with three point vertices

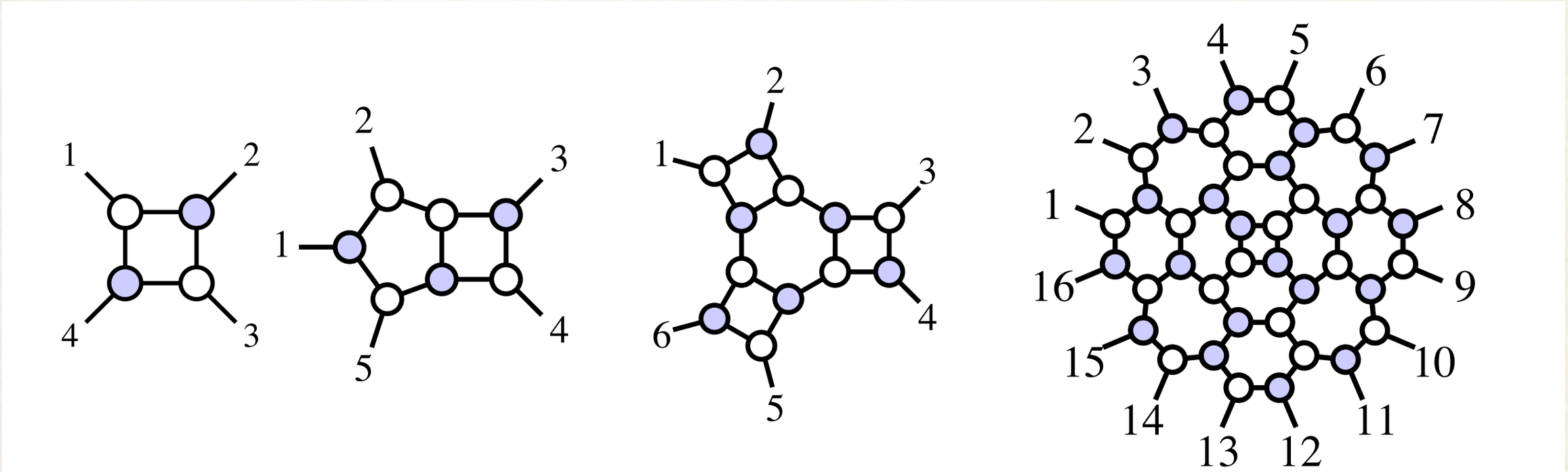


Products of three point amplitudes

$$\begin{cases} P > 4L & \text{Extra delta functions} \\ P = 4L & \text{Function of external data only} \\ P < 4L & \text{Unfixed parameters (forms)} \end{cases}$$

On-shell diagrams

- Draw arbitrary graph with three point vertices



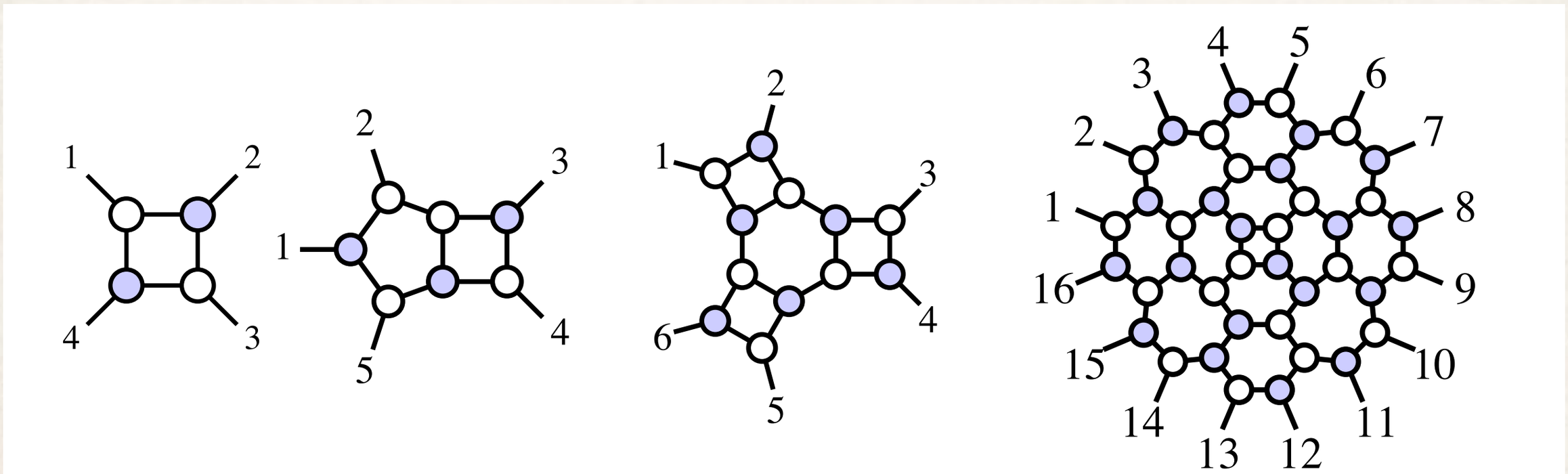
On-shell diagrams with $P \leq 4L$ are cuts of the amplitude

- Parametrized by n, k

$$k = 2B + W - P$$

On-shell diagrams

- Draw arbitrary graph with three point vertices

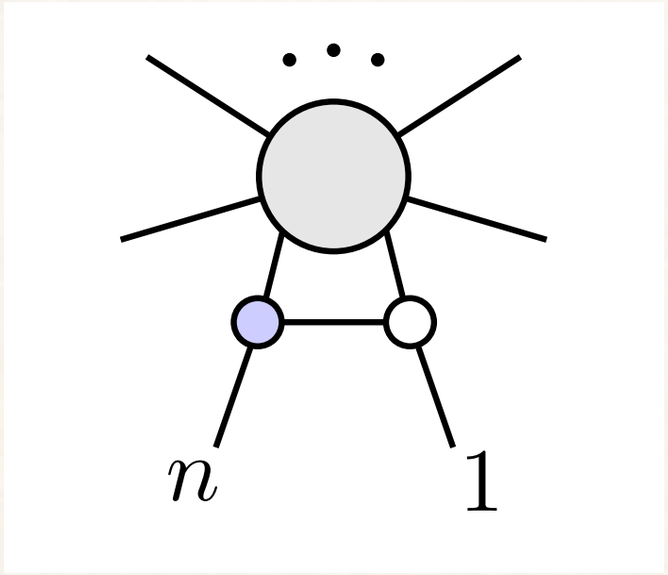


Question: Can we build amplitude from on-shell diagrams?

Recursion relations

BCFW shift

- Consider following diagram



One more loop

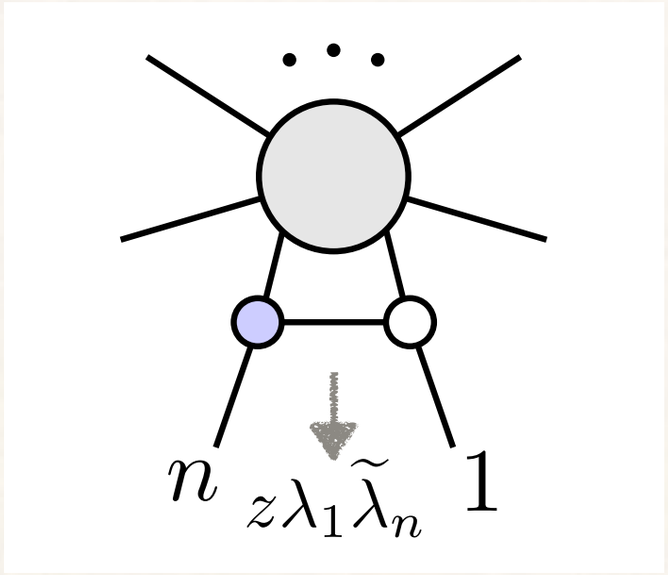
Three more on-shell conditions

↓

Adding one parameter

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

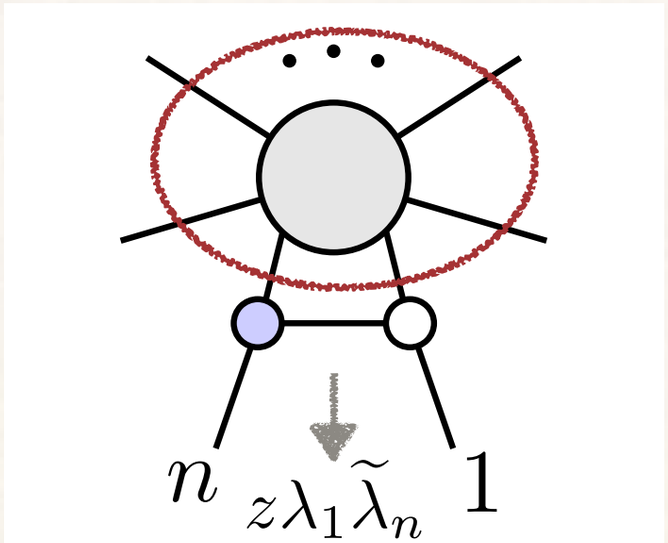
↓

Adding one parameter

New formula: $$K_1(z) = \frac{dz}{z} K_0(z)$$

BCFW shift

- Consider following diagram



One more loop

Three more on-shell conditions

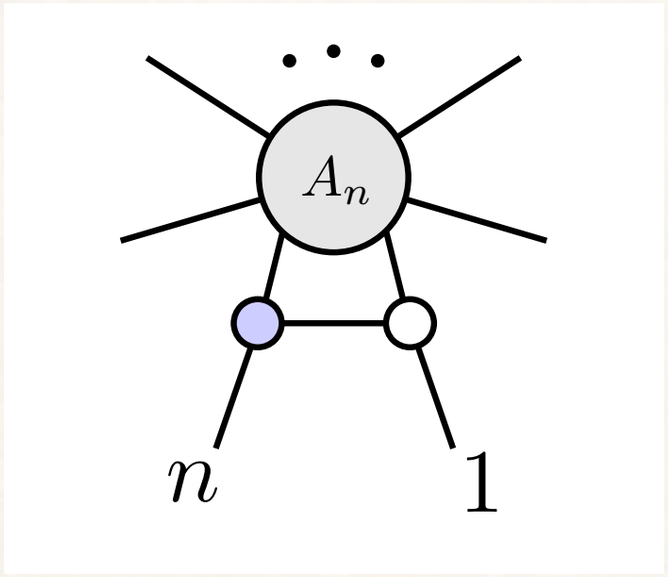
Adding one parameter

New formula: $K_1(z) = \dfrac{dz}{z} K_0(z)$ → Old on-shell diagram with shift

$$\lambda_n \to \lambda_n + z\lambda_1$$
$$\widetilde{\lambda}_1 \to \widetilde{\lambda}_n - z\widetilde{\lambda}_1$$

BCFW recursion relations

- Suppose the blob is the amplitude



Shifted amplitude $\lambda_n \to \lambda_n + z\lambda_1$

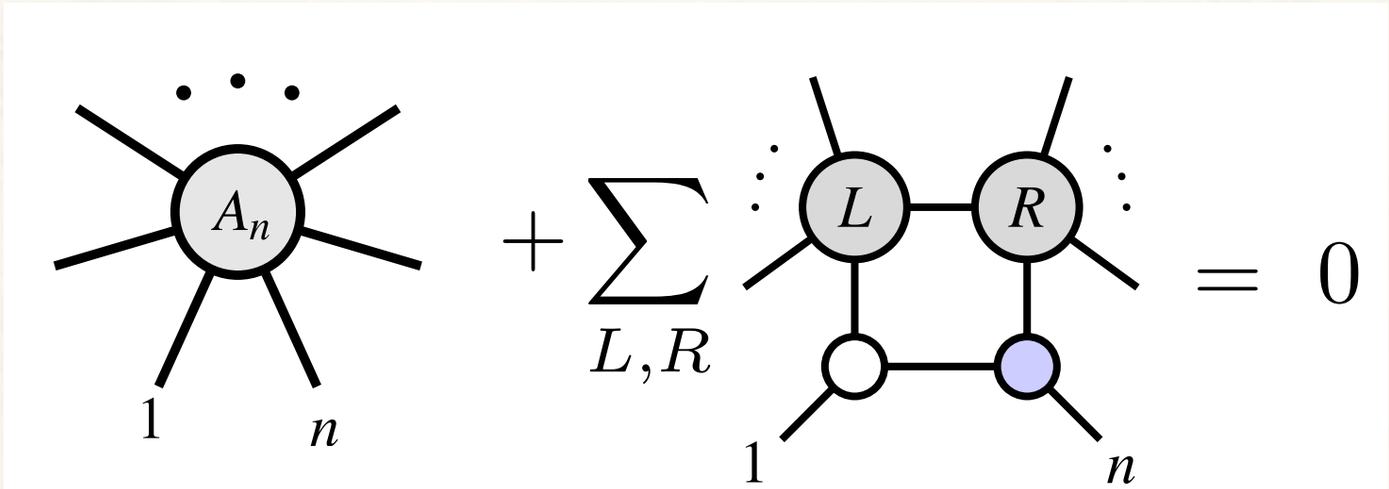
$= \mathcal{A}_n(z)$ $\widetilde{\lambda}_1 \to \widetilde{\lambda}_n - z\widetilde{\lambda}_1$

- Cauchy formula $\partial\, \mathcal{A}_n(z) = 0$

Take the residue on $z = z_k$ $\leftrightarrow$ Erase an edge in the diagram

BCFW recursion relations

- Recursion relations for amplitude

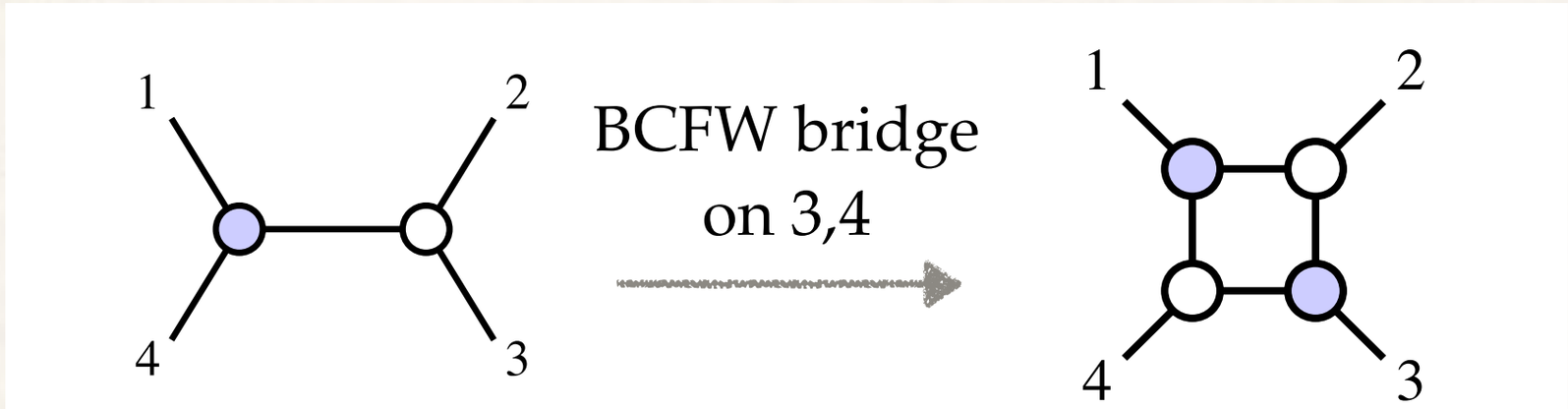


- Tree-level amplitude = sum of on-shell diagrams

- Term-by-term identical to terms in BCFW recursion

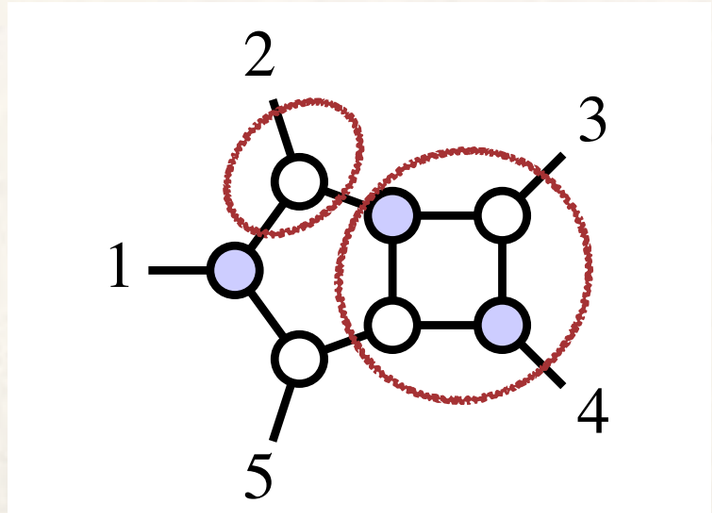
Simple examples

- Four point: only one factorization channel



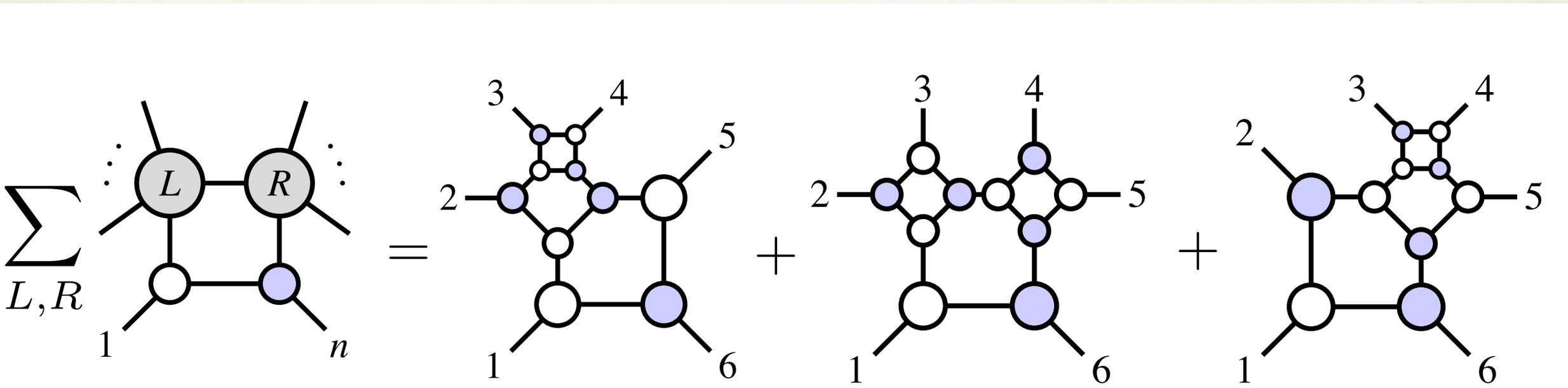
- Five point amplitude

Bridge 5,1 on 3pt
and 4pt amplitudes



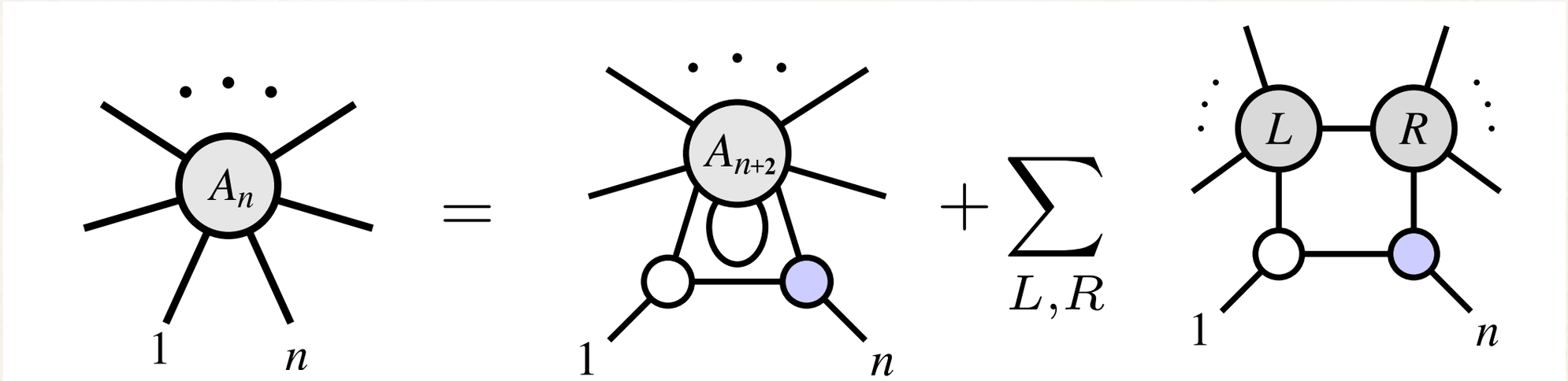
Six point example

- Three diagrams



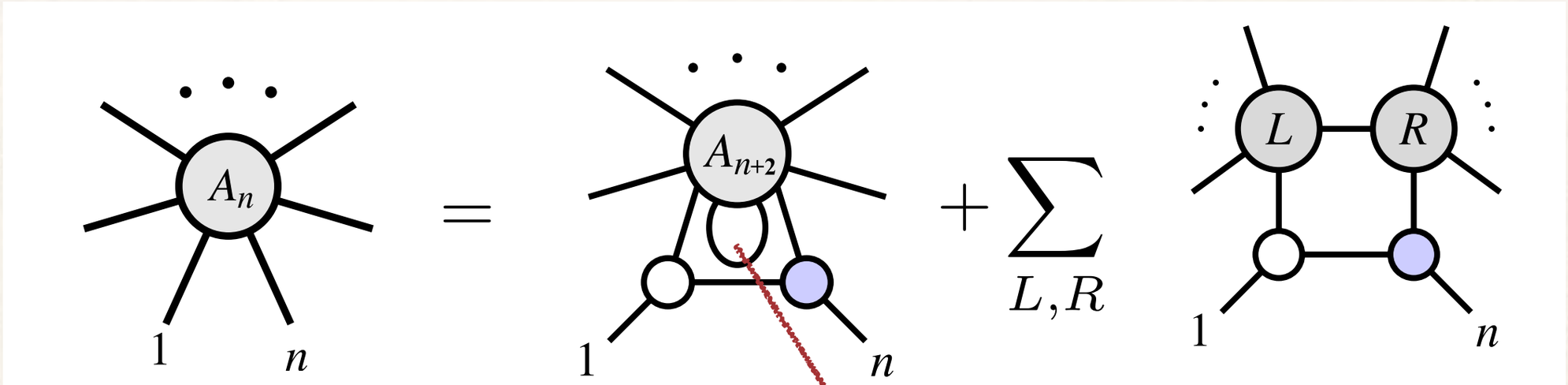
Loop recursion relations

- Recursion relations for ℓ-loop integrand



Loop recursion relations

- Recursion relations for ℓ-loop integrand



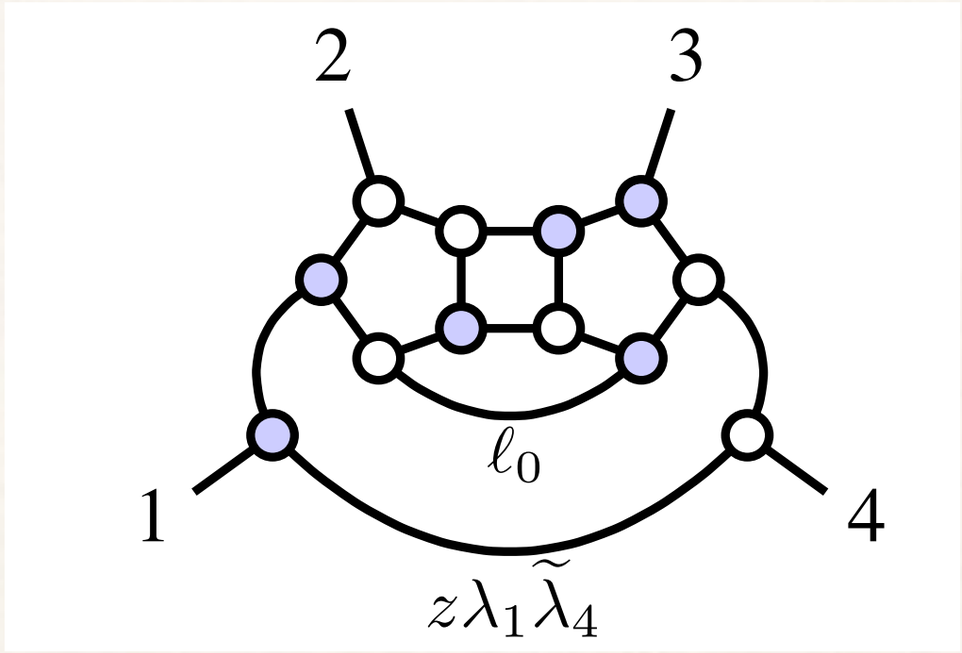
- Loop orders: $(\ell - 1)$ ℓ_1, ℓ_2 $\ell_1 + \ell_2 = \ell$

- New loop momentum $\ell^{(L)} = \ell_0^{(L)} + z\lambda_1\widetilde{\lambda}_n$

$$(\ell_0^{(L)})^2 = 0$$

Four point one loop amplitude

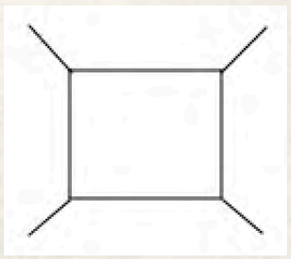
- It is given by one diagram



$$\ell = \ell_0 + z\lambda_1\widetilde{\lambda}_4$$

- 4 complex parameters -> impose reality condition

- 5-loop on-shell diagram = 1-loop off-shell box

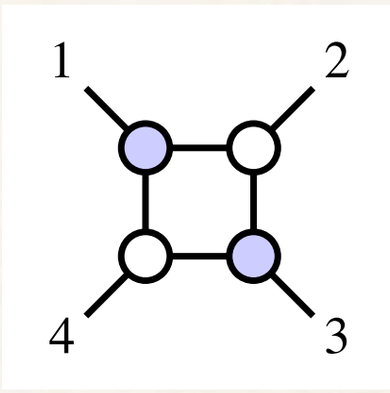


Dimensionality of diagrams

- Tree-level recursion: diagrams with $P = 4L$ contribute

  rational functions of external kinematics

  no delta functions, no free parameters

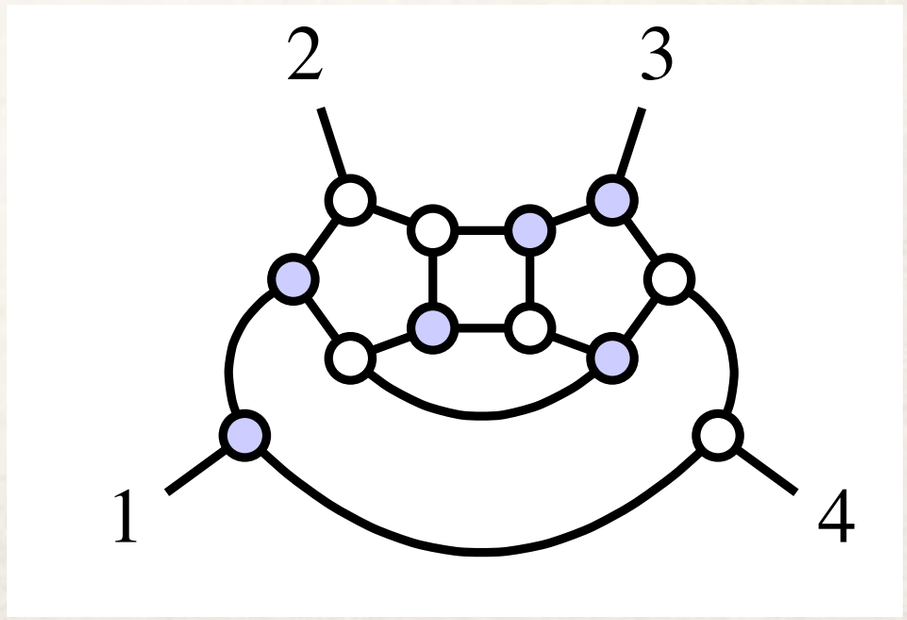


- These are also leading singularities of loop amplitudes

- Loop level: free parameters left

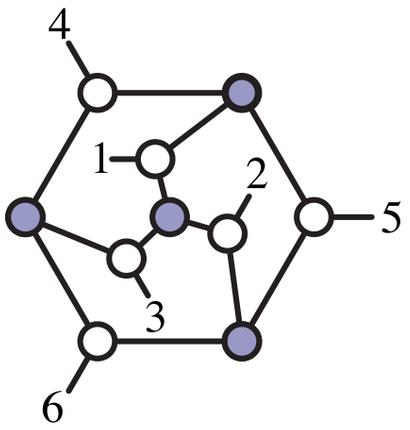
  components of loop momenta

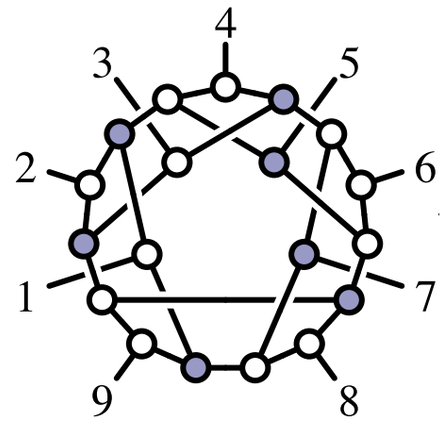
$$\text{free} = 4L - P \qquad \begin{array}{c} P = 16 \\ L = 5 \\ \text{free} = 4 \end{array}$$



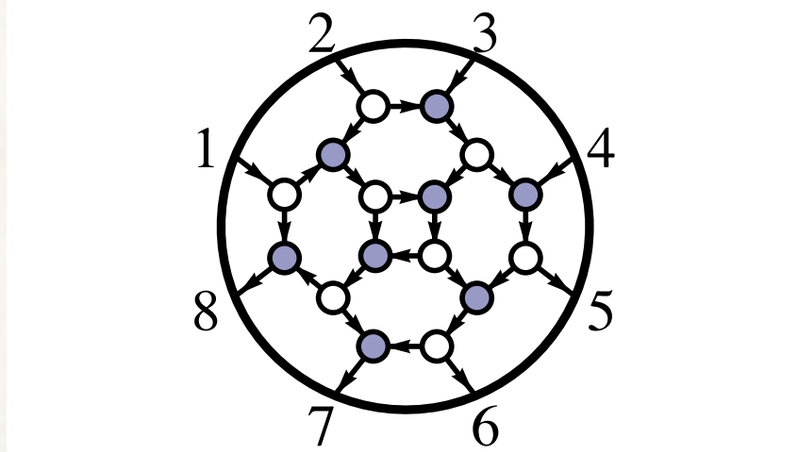
On-shell diagrams in other theories

- On-shell diagrams are well defined in other theories





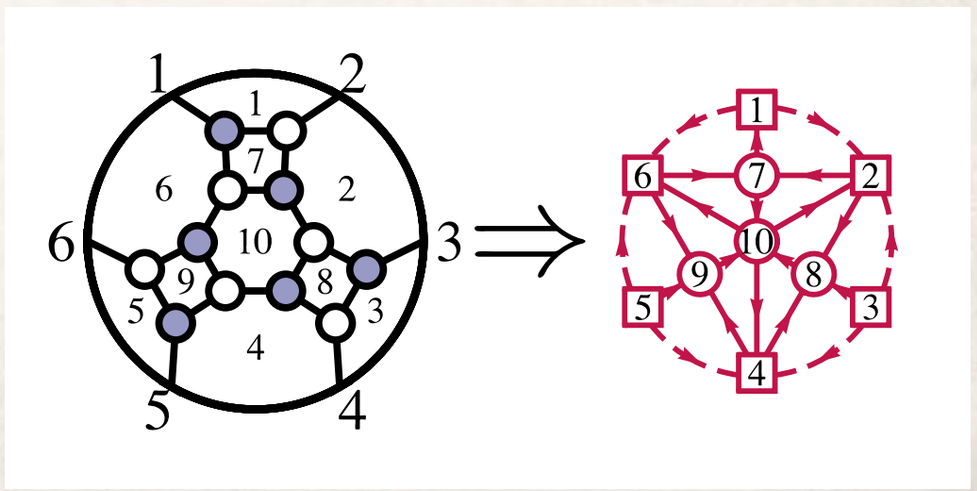
Non-planar N=4 SYM



Pure Yang-Mills

ABJM

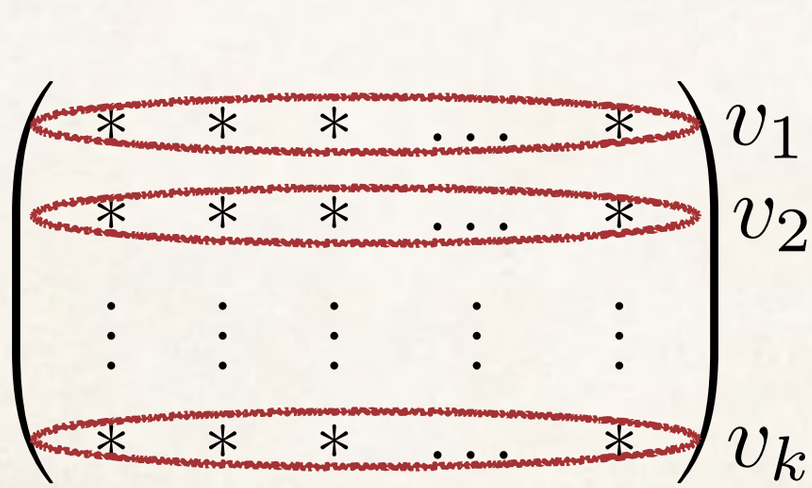
- Relation to quiver gauge theories
- KP equations,….



Positive Grassmannian

Grassmannian

- Describe k-dimensional plane in n-dimensional space
- Choose k basis vectors v_j on this plane



Mod out by GL(k):
Choice of basis vectors

$$k \begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix} \quad n$$

Grassmannian
$G(k,n)$

Positive Grassmannian

- Real space, demand that all main minors are positive

$$k\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}_{n}$$

$$k\begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix}_{k} \geq 0$$

- Positive Grassmannian $G_+(k,n)$
- This is a $k(n-k)$ dimensional space
- Boundaries when minors are zero

Trivial example

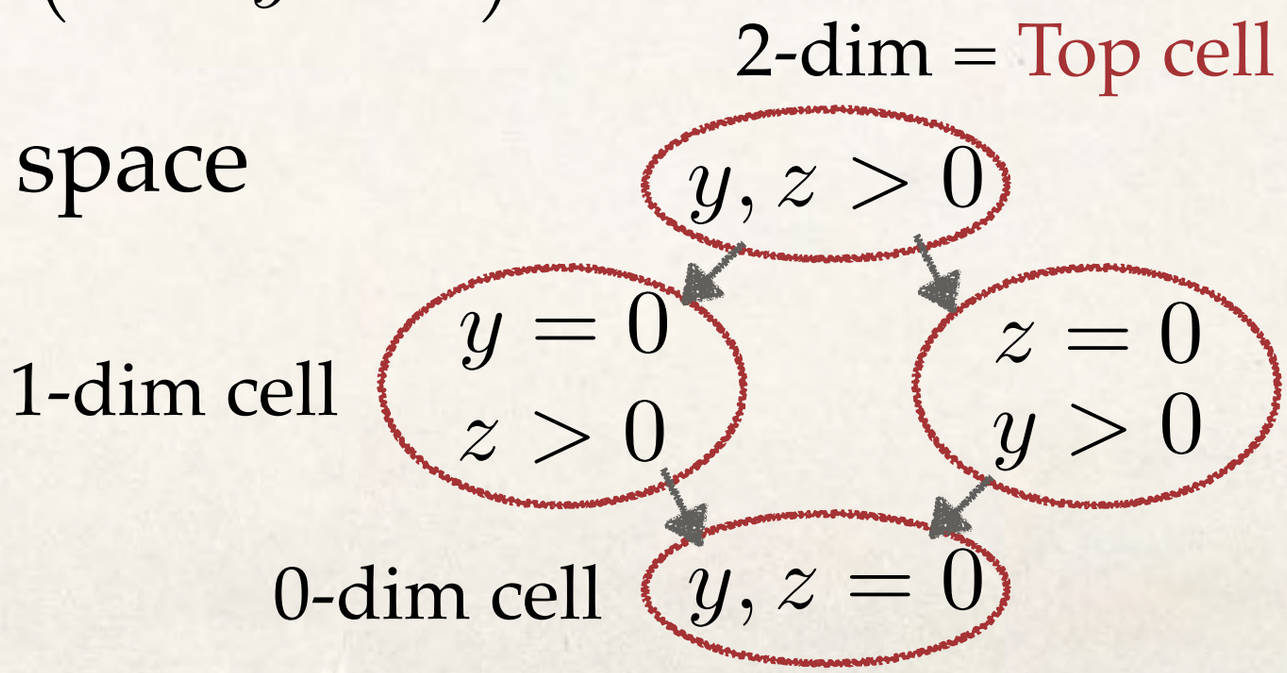
- Let us consider $G_+(1,3)$

$$C = \begin{pmatrix} x & y & z \end{pmatrix} \qquad x, y, z \geq 0$$

- Fix GL(1) transformation: fix one of the parameters

$$C = \begin{pmatrix} 1 & y & z \end{pmatrix}$$

- Two-dimensional space



Simple example

- Let us consider $G_+(2,4)$

$$C = \begin{pmatrix} 1 & a & 0 & -b \\ 0 & c & 1 & d \end{pmatrix} \qquad a, b, c, d \geq 0$$

- This is 4-dimensional space with boundaries

- Top cell: $a, b, c, d > 0$ $\qquad m_{24} = ad + bc$

  Not a boundary

- Example of 3-dim cell: $a = 0 \quad b, c, d > 0$

  etc.

- Example of 2-dim cell: $a = c = 0 \qquad b, d > 0$

Cells of positive Grassmannian

- Cell is specified by the list of vanishing minors

- Top cell $k(n-k)$ - dimensional: nothing vanishes

- Lower dimensional cell: some minors vanish

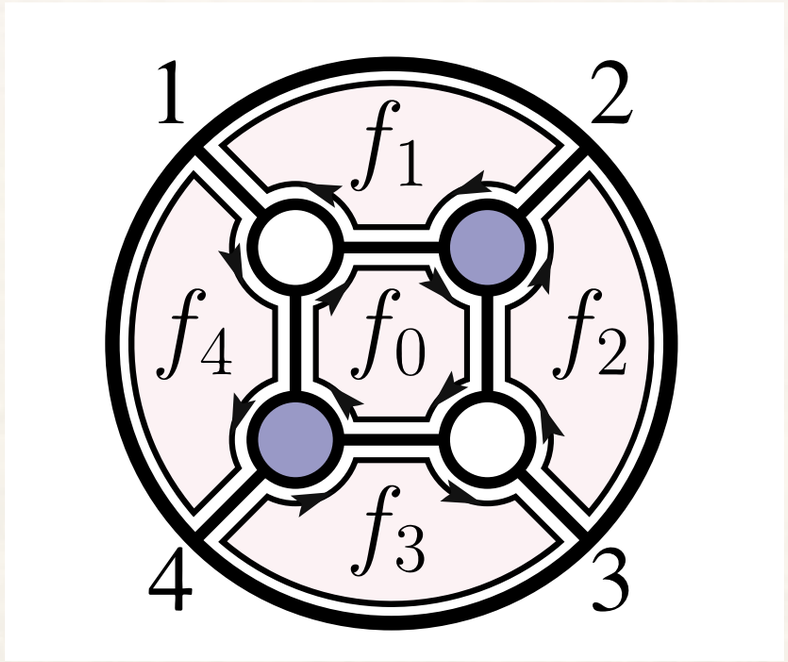
- Parametrization: positive coordinates

$$\text{vanishing minors} \leftrightarrow \text{vanishing coordinates}$$

Back to on-shell diagrams

Face variables

- Draw a graph with two types of three point vertices
- Associate variables with the face of diagram

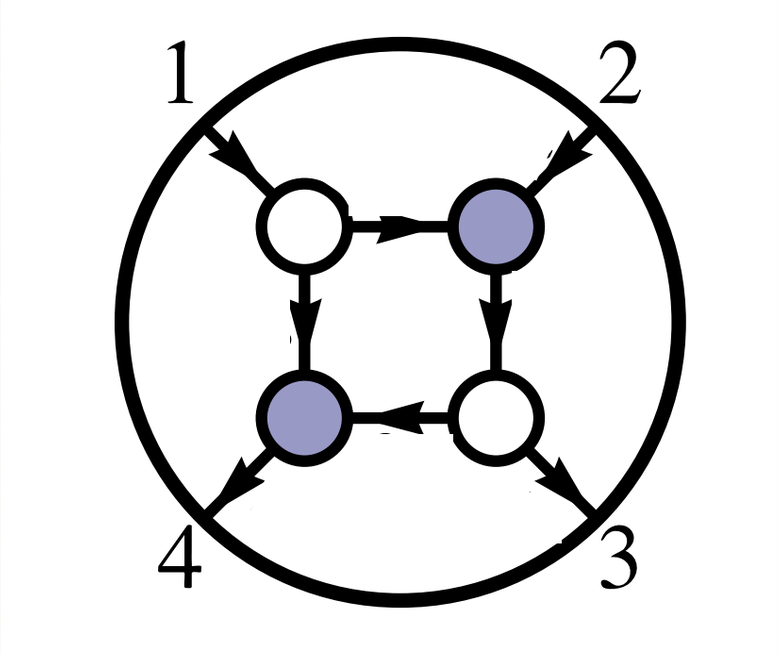


with the property

$$\prod_j f_j = -1$$

Perfect orientation

- Back to on-shell diagrams



Perfect orientation

White vertex: one in, two out

Black vertex: two in, one out

- Not unique, always exists at least one
- Two (k) incoming, two (n-k) outgoing

Entries of matrix

- Define elements of $(k \times n)$ matrix

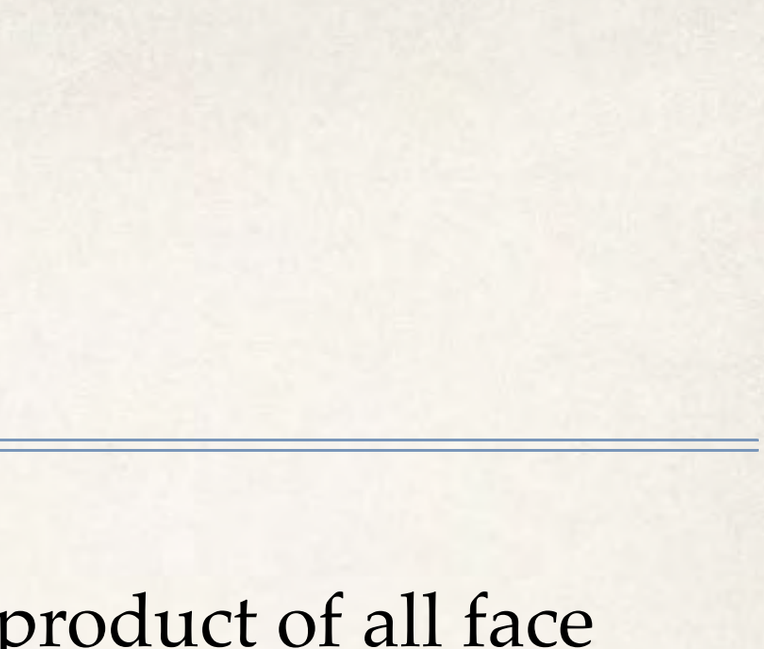
$$c_{ab} = -\sum_{\Gamma} \prod_{j} (-f_j)$$

product of all face variables to the right of the path

incoming

if b incoming
$c_{aa} = 1$
$c_{ab} = 0$

sum over all allowed paths



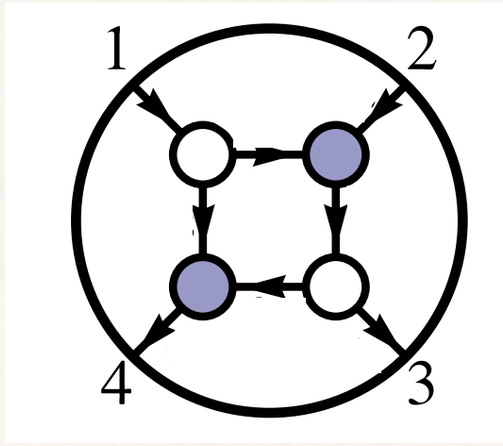
- Example: $c_{11} = c_{22} = 1 \quad c_{12} = c_{21} = 0$

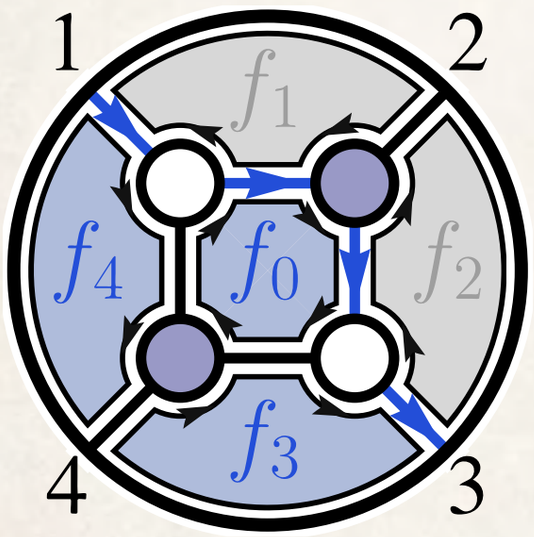
$$c_{13} = *,\ c_{14} = *,\ c_{23} = *,\ c_{24} = *$$

Entries of matrix

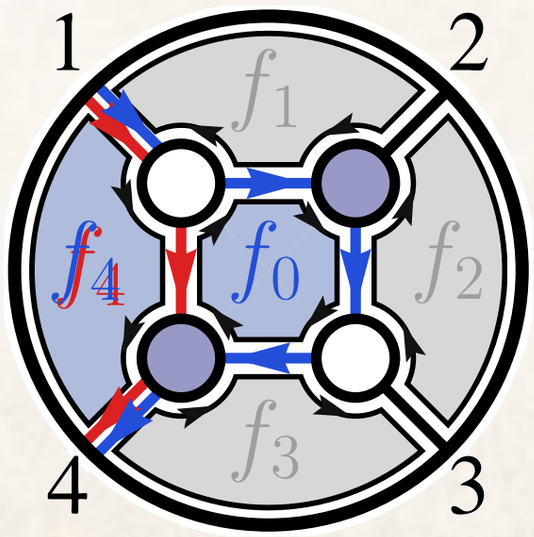
Apply on our example

$$c_{ab} = -\sum_{\Gamma} \prod_{j} (-f_j)$$

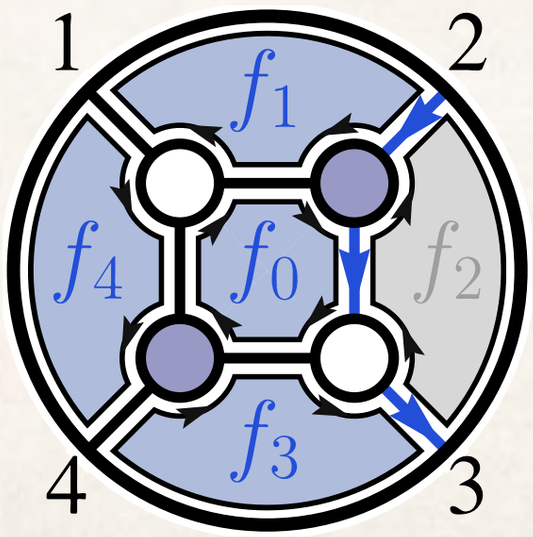




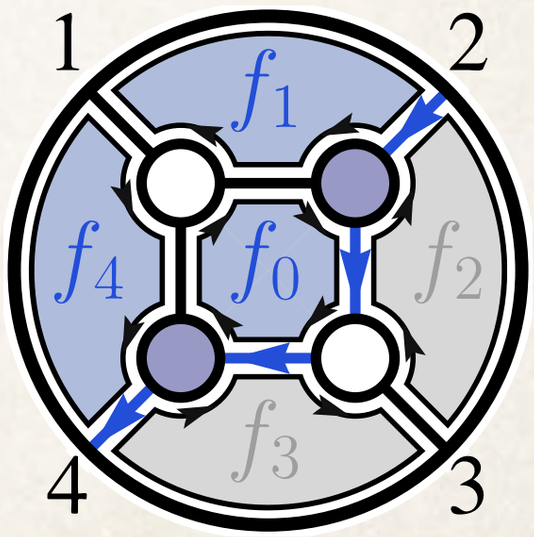
$-c_{13} = -f_0\, f_3\, f_4$



$-c_{14} = f_0\, f_4 - f_4$



$-c_{23} = f_0\, f_1\, f_3\, f_4$



$-c_{24} = f_0\, f_1\, f_4$

Entries of matrix

- The matrix is

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

f_2 eliminated

- There always exists choice of signs for f_i such that

$$C \in G_+(k,n)$$

- For our case:

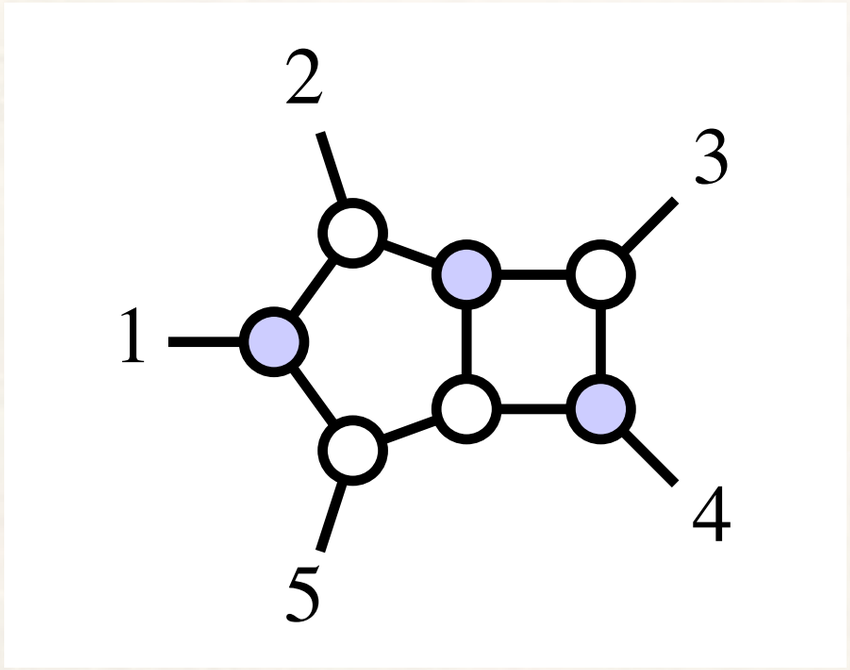
$$m_{12} = 1 \qquad m_{23} = -f_0 f_3 f_4$$
$$m_{13} = -f_0 f_1 f_3 f_4 \qquad m_{24} = -f_4(1-f_0)$$
$$m_{14} = -f_0 f_1 f_4 \qquad m_{34} = f_0 f_1 f_3 f_4^2$$

$\rightarrow$ $f_0 < 0$, $f_1 < 0$, $f_3 > 0$, $f_4 < 0$

All minors positive

Positive Grassmannian from on-shell diagram

- On-shell diagram: method how to generate $C \in G_+(k,n)$



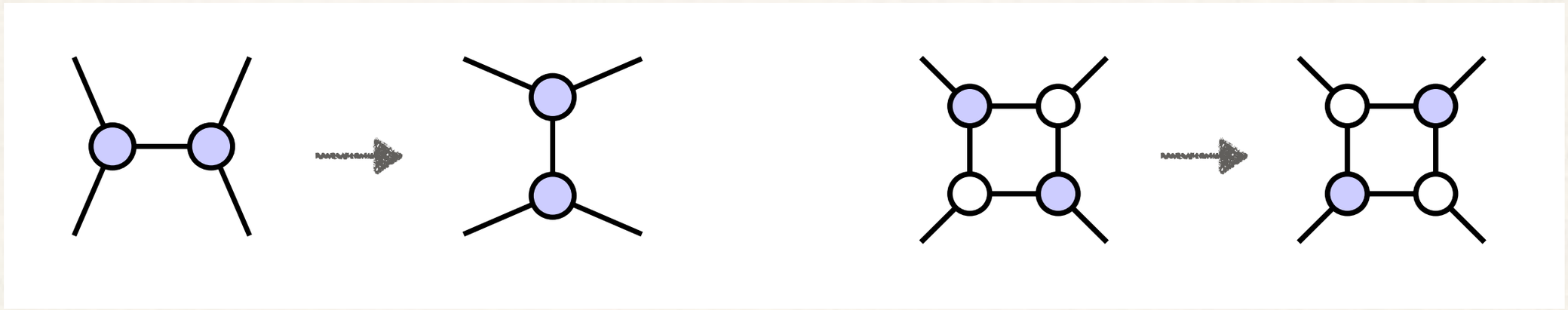
$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

$$\begin{vmatrix} * & * \\ * & * \end{vmatrix} \geq 0$$

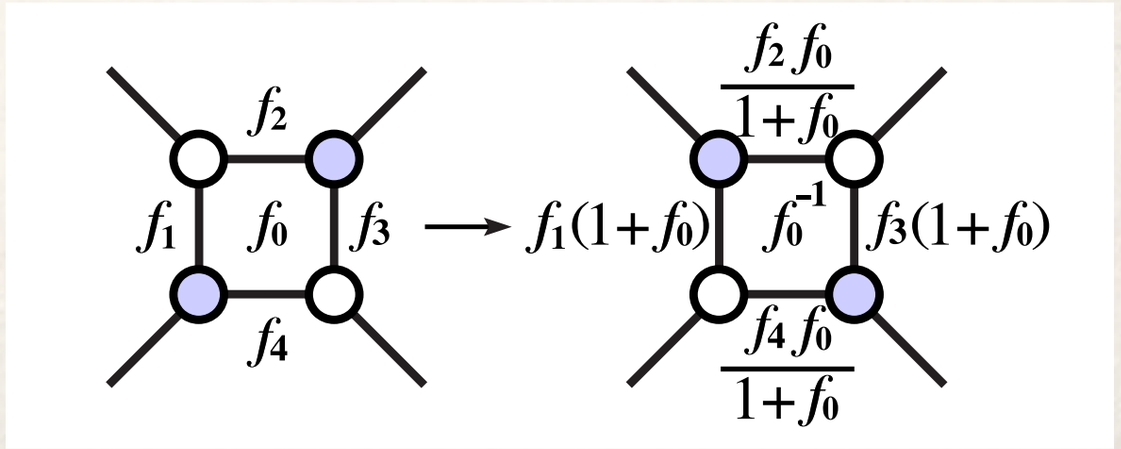
- All cells can be generated using on-shell diagrams

Identity moves

- The on-shell diagram is not unique for the cell
- Identity moves anywhere in the diagram

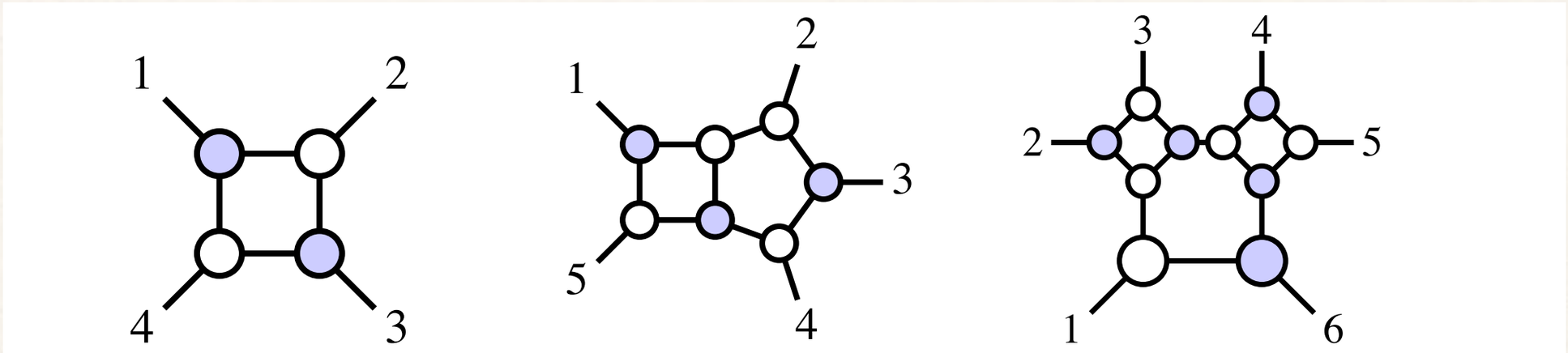


- This preserves the cell in positive Grassmannian



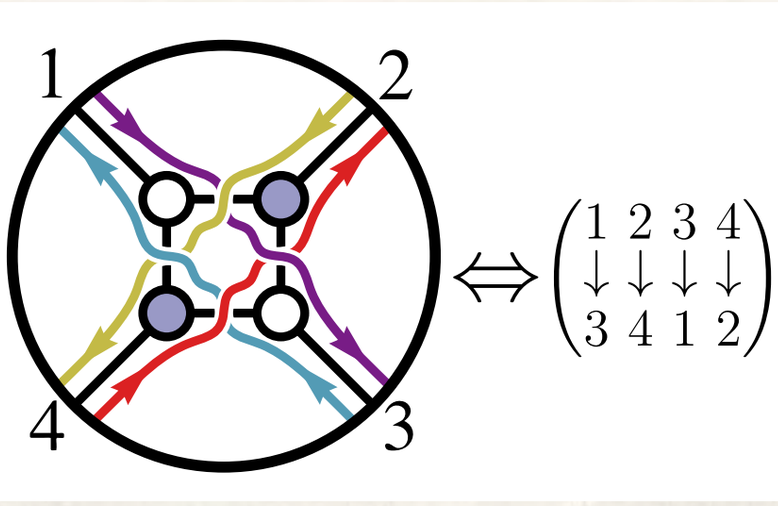
Permutations

- Set of on-shell diagrams with P=4L: Reduced diagrams



- Isomorphic to permutation of n indices $n \to \sigma(n)$
  (modulo identity moves)

- Example:



In recursion for tree-level amplitudes

It is very interesting that the same objects
appear in physics and mathematics.

But is it useful for something?



Grassmannian Geometry of Scattering Amplitudes

LECTURE 3

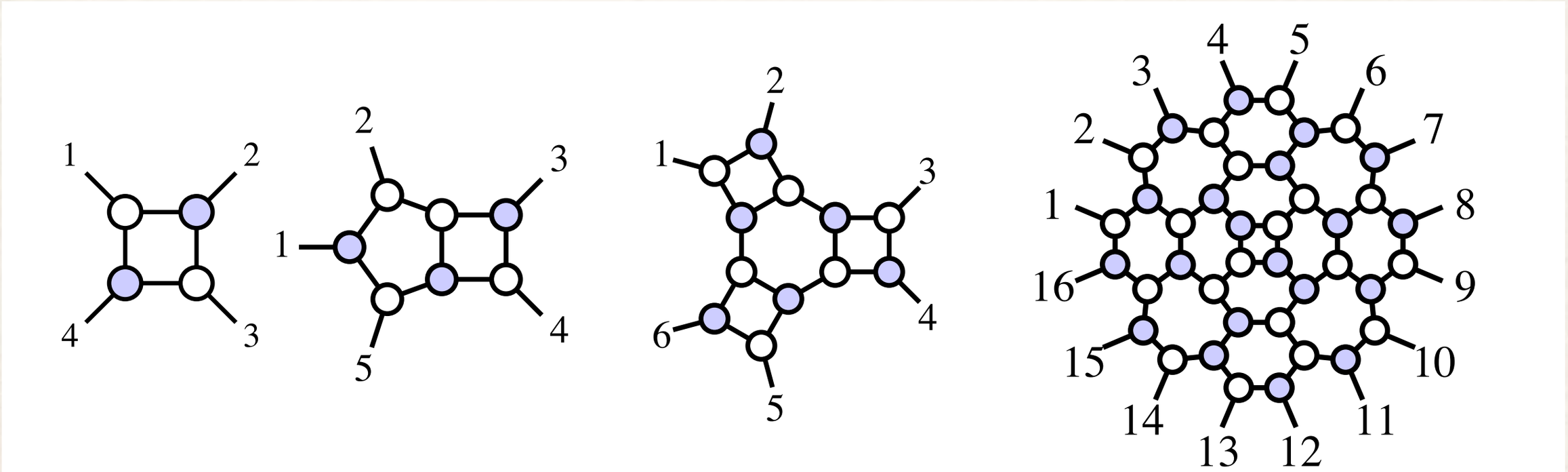
Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP), UC Davis

Amplitudes School, Edinburgh, July 2017

On-shell diagrams

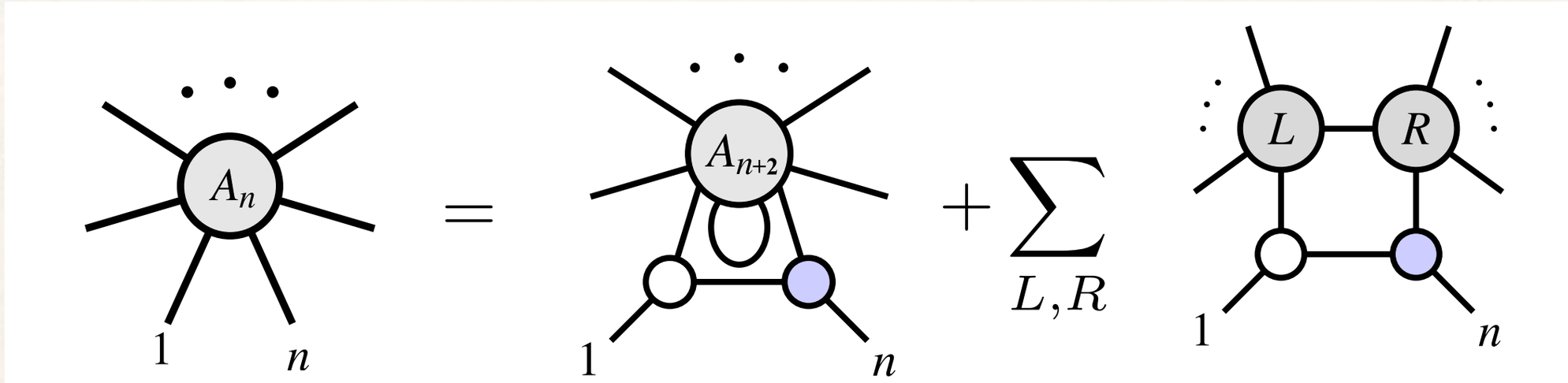
- Draw arbitrary graph with three point vertices



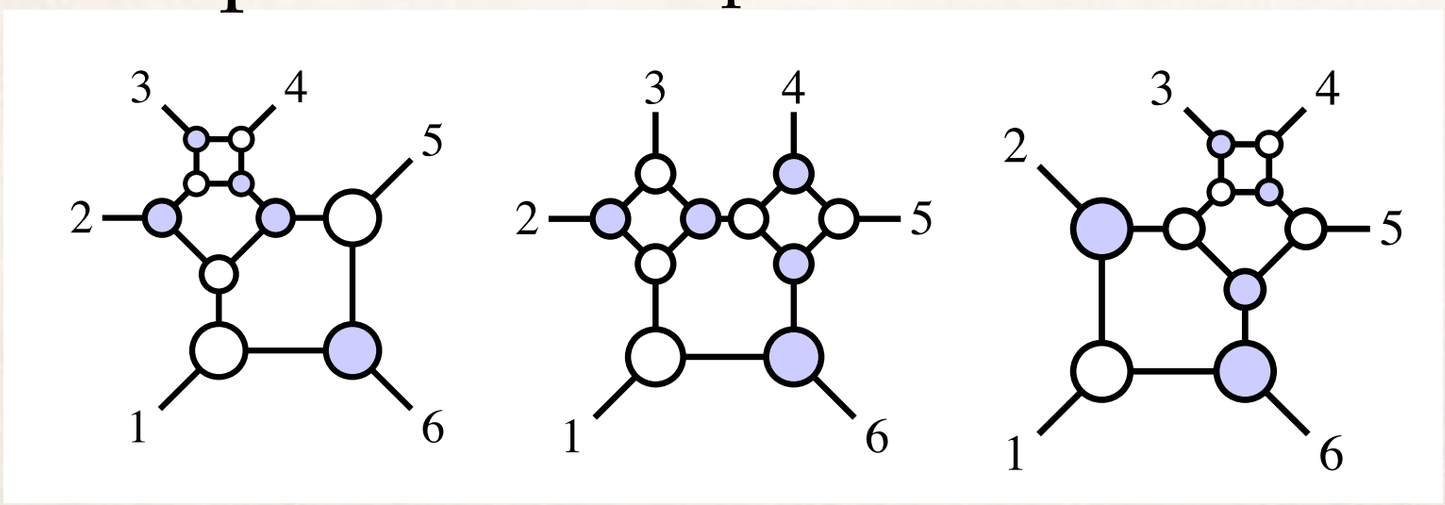
Product of three point on-shell amplitudes

Recursion relations

- Recursion relations for ℓ-loop integrand

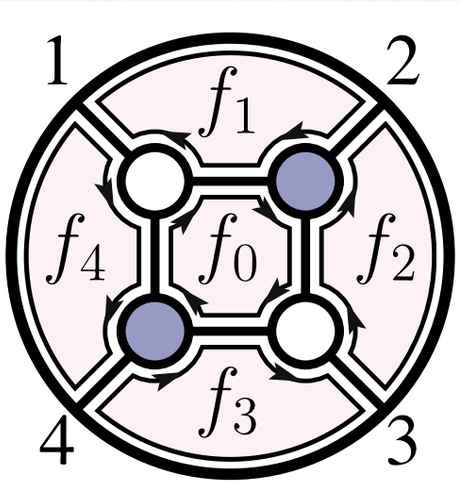


- Examples: 6pt tree 4pt 1-loop



Positive Grassmannian

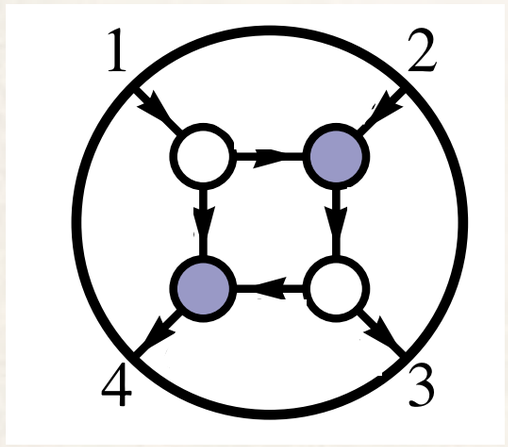
- Face variables



with the property

$$\prod_j f_j = -1$$

- Perfect orientation



Elements of $(k \times n)$ matrix

$$c_{ab} = -\sum_{\Gamma} \prod_j (-f_j)$$

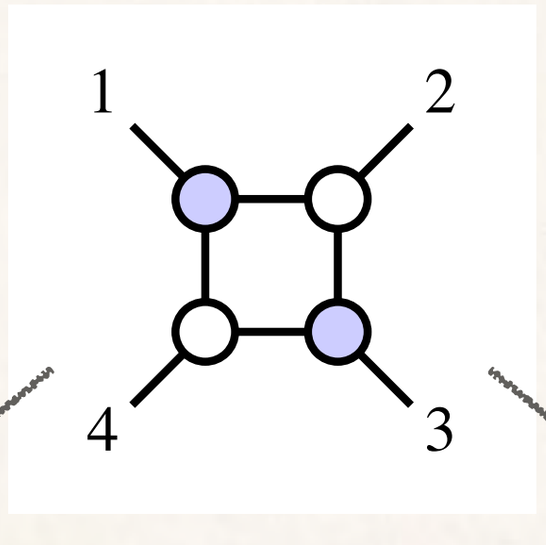
$$\overset{n}{k\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}}$$

Cell in positive Grassmannian $G_+(k,n)$

$$\overset{k}{k\begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix}} \geq 0$$

Physics from Grassmannian

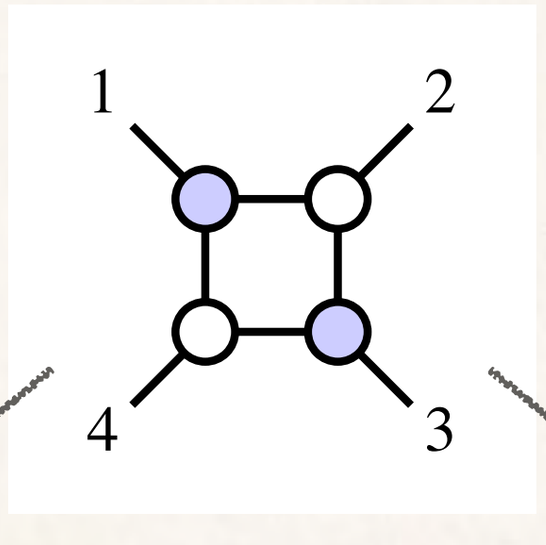
Connection for N=4 SYM



$$R = \mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\mathcal{M}_3^{tree}\mathcal{M}_4^{tree}$$

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

Connection for N=4 SYM



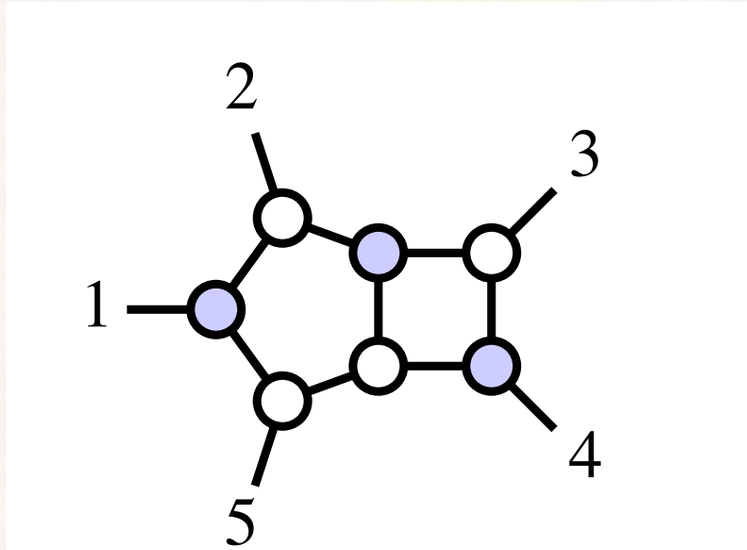
$$R = \mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\mathcal{M}_3^{tree}\mathcal{M}_4^{tree}$$

$$C = \begin{pmatrix} 1 & 0 & f_0 f_3 f_4 & f_4(1-f_0) \\ 0 & 1 & -f_0 f_1 f_3 f_4 & -f_0 f_1 f_4 \end{pmatrix}$$

$$R = \int \frac{df_0}{f_0}\frac{df_1}{f_1}\frac{df_2}{f_2}\frac{df_3}{f_3}\delta(C\cdot Z) \qquad Z = (\lambda, \widetilde{\lambda}, \widetilde{\eta})$$

Dual picture for on-shell diagrams

For arbitrary
on-shell diagram



- Label face variables
- Find perfect orientation
- Construct the Grassmannian matrix
- Write a logarithmic form

$$R = \int \frac{df_1}{f_1}\frac{df_2}{f_2}\ldots\frac{df_m}{f_m}\,\delta(C\cdot Z) \leftrightarrow \mathcal{M}_1^{tree}\mathcal{M}_2^{tree}\ldots\mathcal{M}_m^{tree}$$

Momentum conservation

- Very good reason for the Grassmannian
- Linearize momentum conservation

$$\delta(P) = \delta\left(\sum_a \lambda_a \widetilde{\lambda}_a\right)$$

- We want to write it as two linear factors

$$\delta\left(C_{ab}\widetilde{\lambda}_b\right)\ \delta\left(D_{ab}\lambda_b\right)$$

Grassmannian

- You get k relations, (k x n) matrix C / GL(k)

$$\delta\left(C_{ab}\widetilde{\lambda}_b\right) \quad \begin{matrix} a=1,\ldots,k \\ b=1,\ldots,n \end{matrix}$$

$$C \in G(k,n)$$

k-plane in n dimensions

$$\delta(D_{ab}\lambda_b) = \delta\left(C^{\perp}_{ab}\widetilde{\lambda}_b\right)$$

- This matrix C has many free parameters: many ways how linearize momentum conservation

- Each on-shell diagram gives you one

Logarithmic form

$$R = \int \frac{df_1}{f_1}\frac{df_2}{f_2}\dots\frac{df_m}{f_m}\,\delta(C\cdot Z)$$

$\alpha = 1,\dots,k$
$\beta = 1,\dots,n$
$\gamma = 1,\dots,n-k$
$a,\dot{a} = 1,2$
$A = 1,\dots,4$

- Delta functions

$$\delta(C\cdot Z) \equiv \delta(C_{\alpha\beta}\lambda^a_\beta) \times \delta(C^\perp_{\gamma\beta}\widetilde{\lambda}^{\dot{a}}_\beta) \times \delta(C_{\alpha\beta}\widetilde{\eta}^A_\beta)$$

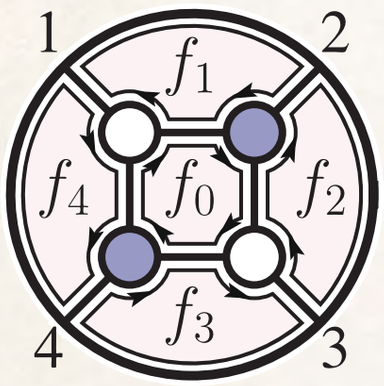
$C^\perp$
orthogonal matrix

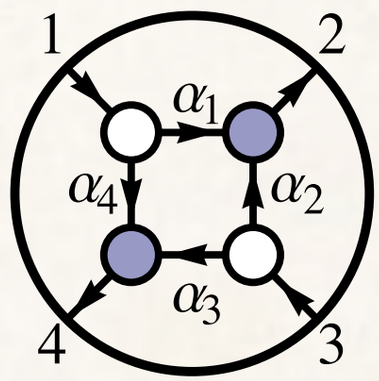
$$k\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}^{n} \longrightarrow n-k\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}^{n}$$

Reproduces momentum conservation $\delta(\lambda\widetilde{\lambda})\delta(\widetilde{\eta}\lambda)$ + Solves for some/all parameters f_j + Gives super delta functions $\delta(\lambda,\widetilde{\lambda},\widetilde{\eta})$

Face variables

- Building matrix: face or edge variables





$$C = \begin{pmatrix} 1 & \alpha_1 & 0 & -\alpha_4 \\ 0 & \alpha_2 & 1 & \alpha_3 \end{pmatrix}$$

- Exciting connection to mathematics

  Choose $\alpha_i > 0$: positive minors -> Positive Grassmannian

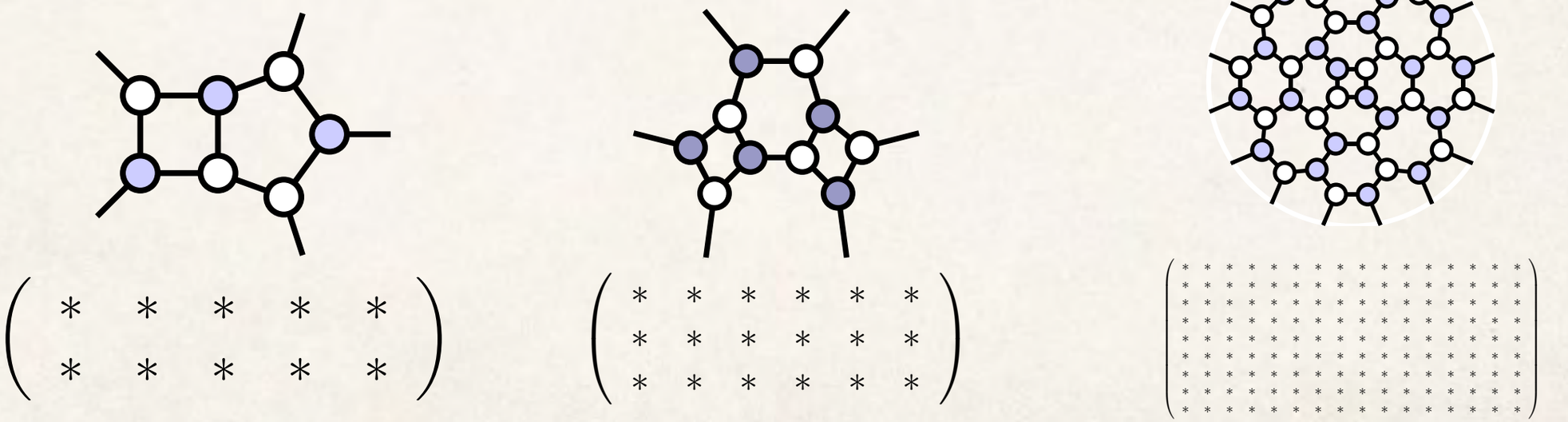
Amalgamation procedure

- Construct big positive matrix from small ones



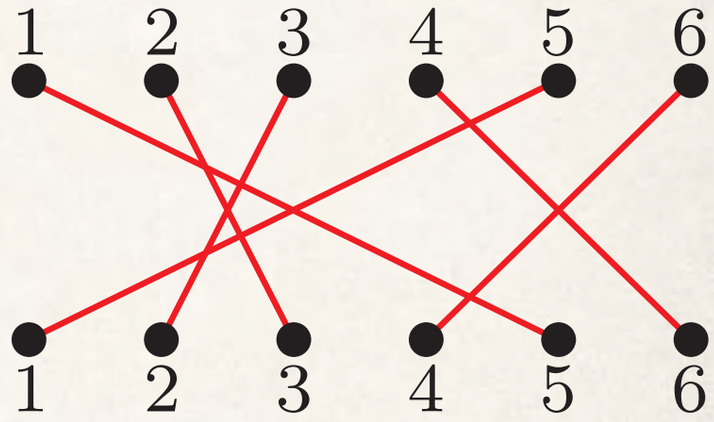
Gluing preserves positivity of minors

- Arbitrary graph: positive matrix

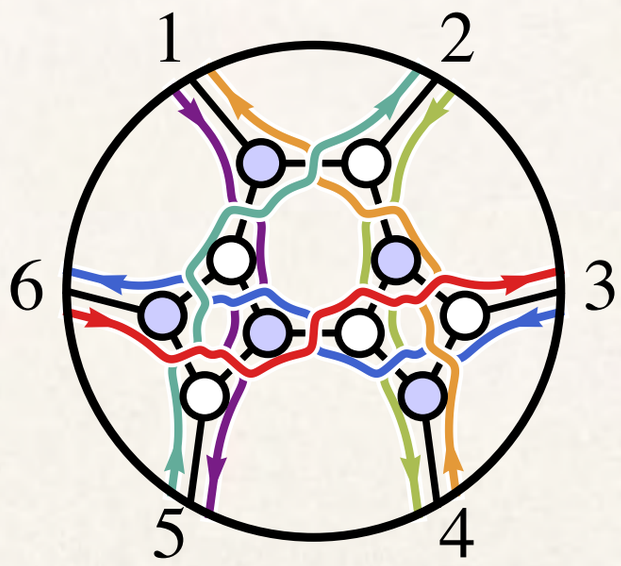


Permutations

- Represent graphically permutation



- Graph with only 3pt vertices
  - Turn right on blue
  - Turn left on white



Tree-level amplitudes: list of permutations

Logarithmic singularities

- Amplitude = sum of on-shell diagrams

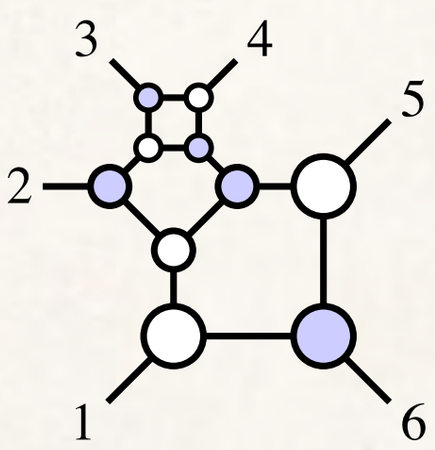
$$\Omega \sim \frac{d\alpha}{\alpha} \quad \text{near any pole} \quad \alpha = F(\ell_j, p_i)$$

- More than single poles: $\dfrac{dx\,dy}{xy(x+y)} \xrightarrow{x=0} \dfrac{dy}{y^2}$

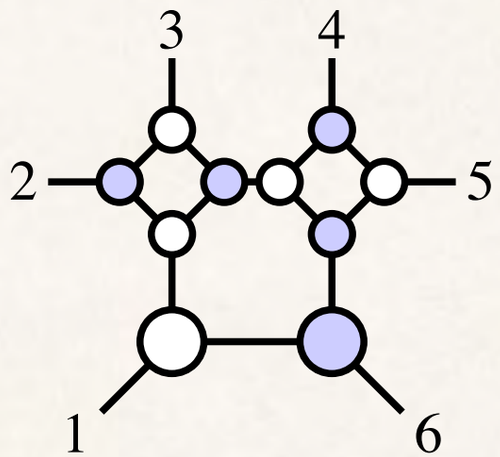
Back to amplitude

- BCFW recursion relation

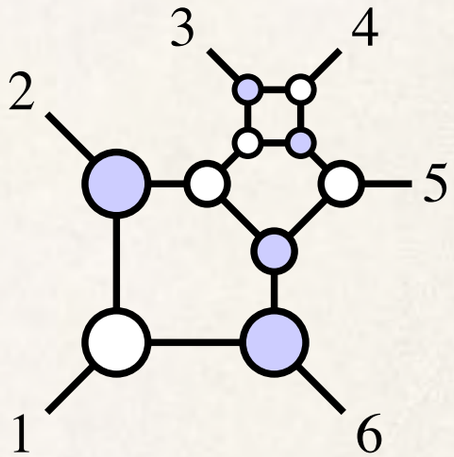
$$\mathcal{M}_6^{tree} =$$



$$+$$



$$+$$



$$\begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} \qquad \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}$$

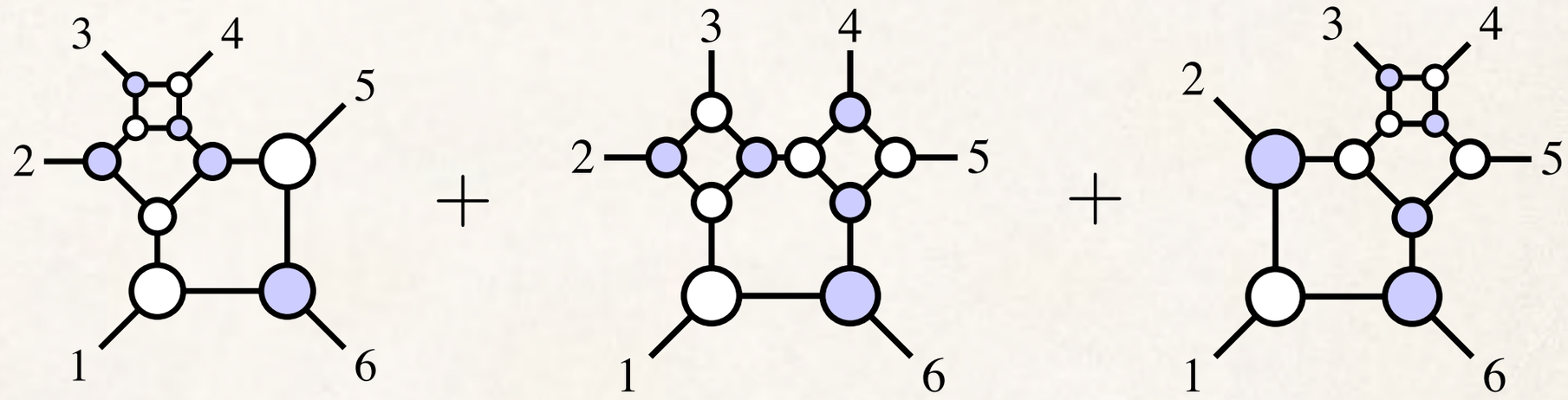
$$m_{561} = 0 \qquad m_{123} = 0 \qquad m_{345} = 0$$

invariant information

Back to amplitude

- BCFW recursion relation

$$\mathcal{M}_6^{tree} =$$



- Physics of on-shell diagrams removed

  Only present in kinematical delta functions

- New mathematical structures play crucial role

Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

N=4 SYM:

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\ldots\frac{d\alpha_n}{\alpha_n}\delta(C\cdot Z)$$

Definition of the theory

- Why is this for N=4 SYM? What about other theories?
- Diagrams and connection to Grassmannian is general
- Specific for theory: differential form

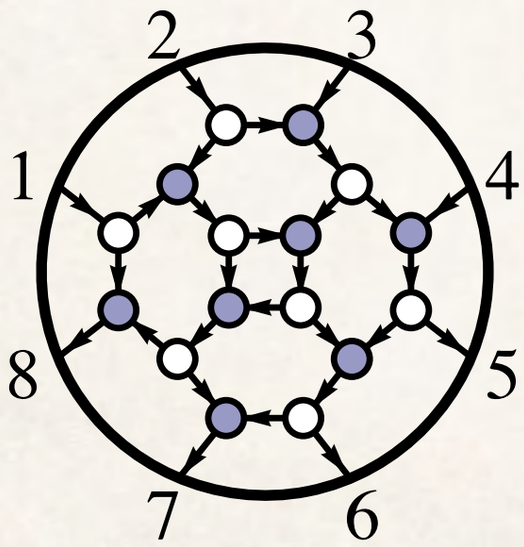
General QFT: $$\Omega = F(\alpha)\,\delta(C \cdot Z)$$

- In a sense $F(\alpha)$ defines a theory (as Lagrangian does)

Exploring space of theories

- One step at a time away from N=4 SYM
- **Case 1**: planar N<4 SYM (including QCD)

(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

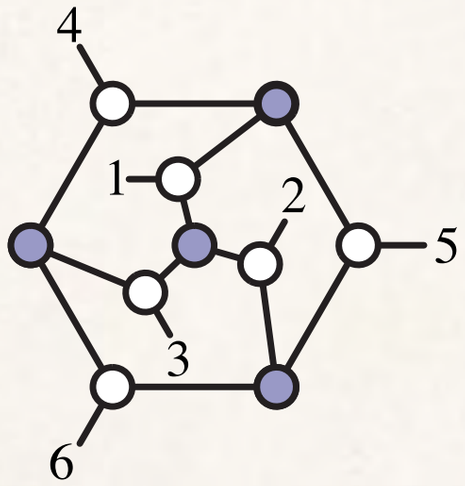


$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\cdots\frac{d\alpha_n}{\alpha_n}\cdot\mathcal{J}(\alpha)\delta(C\cdot Z)$$

Exploring space of theories

- **Case 2:** Non-planar N=4 SYM

(Arkani-Hamed, Bourjaily, Cachazo, Postnikov, JT, 2014) (Bourjaily, Franco, Galloni, Penante, Wen 2015, 2016)



same form as planar

$$\Omega = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\cdots\frac{d\alpha_n}{\alpha_n}\delta(C\cdot Z)$$

less is known on mathematical side

- **Case 3**: N=8 Supergravity

(Herrmann, JT 2016) (Heslop, Lipstein 2016)

Both have natural extensions to lower SUSY

$$\Omega = \frac{d\alpha_1}{\alpha_1^3}\frac{d\alpha_2}{\alpha_2^3}\cdots\frac{d\alpha_m}{\alpha_m^3}\prod_v \Delta_v\cdot\delta(C\cdot Z)$$

Connection to amplitude

- At the moment no loop recursion relations
  - We can not construct the amplitude
  - We do not know if properties of on-shell diagrams survive
- Some encouraging results for full N=4 SYM

(Arkani-Hamed, Bourjaily, Cachazo, JT, 2014) (Bern, Herrmann, Litsey, Stankowicz, JT 2014, 2015)

  - Implications of possible existence of recursion relations
  - New features do not follow from known symmetries

New hidden symmetry?

Summary

- On-shell diagrams: gauge invariant on-shell objects

- Exist in any weakly coupled QFT

- Cuts of loop amplitudes + building blocks in BCFW recursion relations

- Connection to positive Grassmannian - physics derived from geometry!

Hydrogen atom of gauge theories: planar N=4 SYM theory

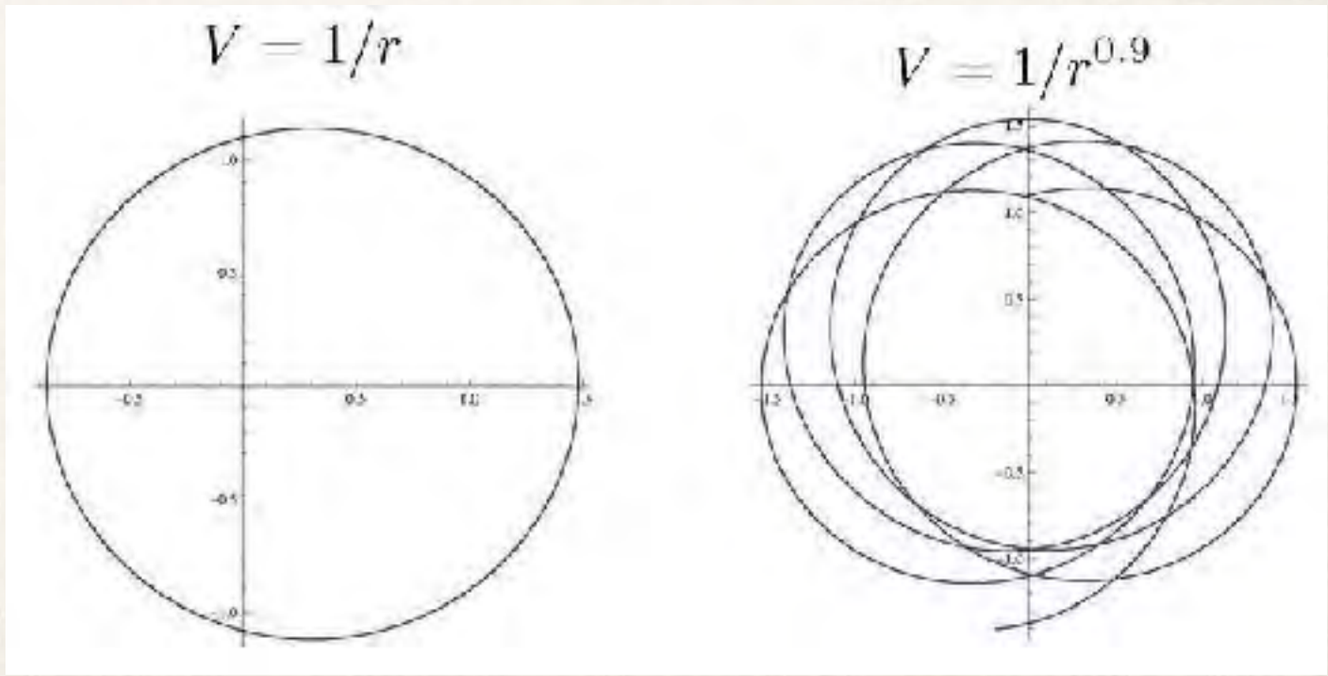
Toy models

- Hard to make progress on difficult questions in full generality: time-proven method - choose toy model

- Long history of “integrable models”: exactly solvable

- **Kepler problem:**
  - orbits do not precess
  - Runge-Lenz vector

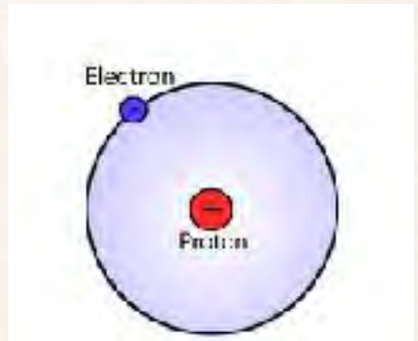
$$\vec{A} = \frac{1}{2}\left(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}\right) - \mu\frac{\lambda}{4\pi}\frac{\vec{x}}{|x|}$$



Toy models

- **Hydrogen atom:** $H = \frac{1}{2m}p^2 - \frac{k}{r}$

  - Hidden symmetry: Runge-Lenz-Pauli vector
  - Allows to find spectrum

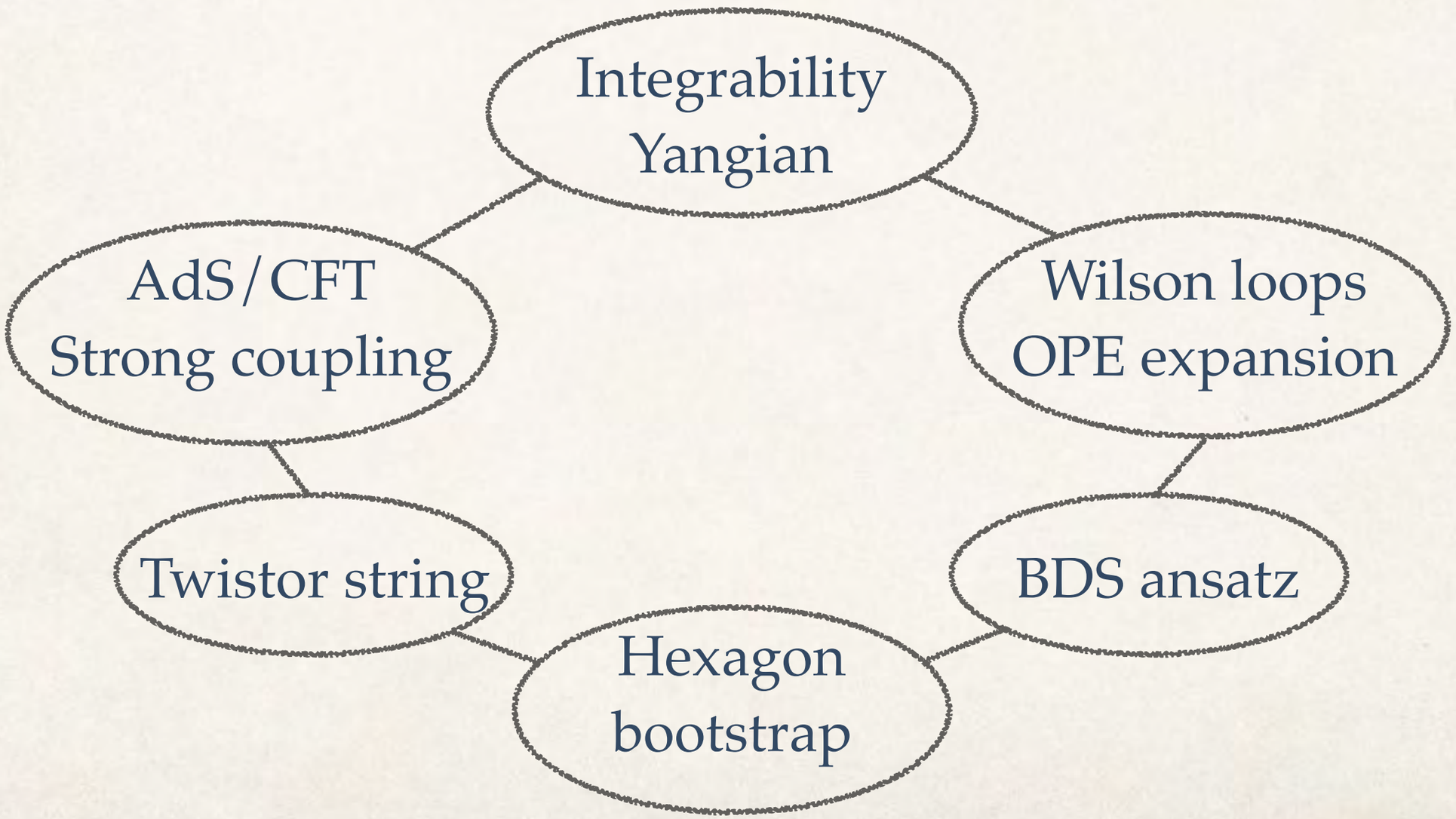


- Toy model for QFT: **planar N=4 SYM theory**

  - Theory of quarks and gluons, similar to QCD but no confinement
  - Hidden symmetry: Yangian - connection to 2d integrable models
  - Great theory to test new ideas in QFT

Many faces of the theory

- Useful playground for many theoretical ideas



Amplitudes in N=4 SYM

- N=4 superfield

$$\Phi = G_+ + \tilde{\eta}_A \Gamma_A + \frac{1}{2}\tilde{\eta}^A \tilde{\eta}^B S_{AB} + \frac{1}{6}\epsilon_{ABCD}\tilde{\eta}^A \tilde{\eta}^B \tilde{\eta}^C \overline{\Gamma}^D + \frac{1}{24}\epsilon_{ABCD}\tilde{\eta}^A \tilde{\eta}^B \tilde{\eta}^C \tilde{\eta}^D G_-$$

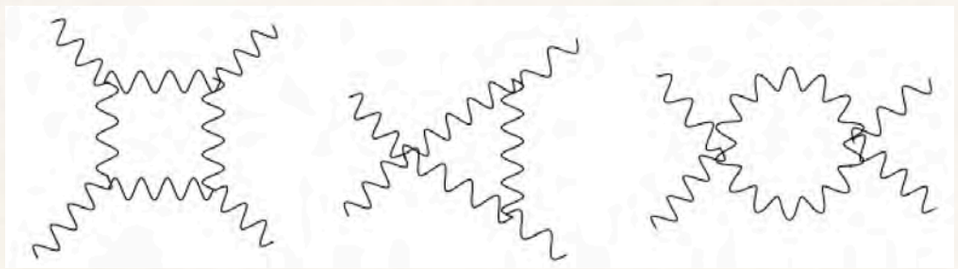
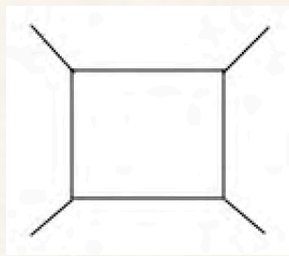
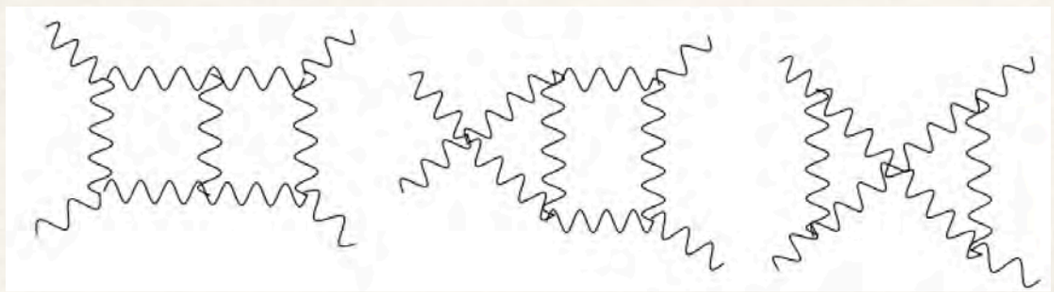
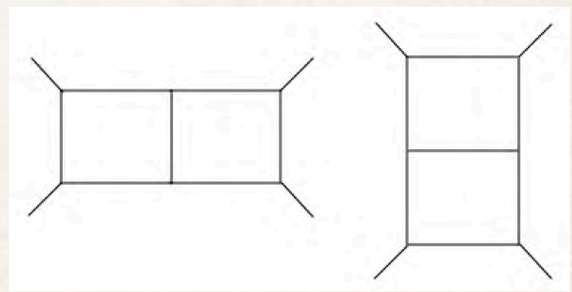
- Superamplitudes: $\mathcal{A}_n = \sum_{k=2}^{n-2} \mathcal{A}_{n,k}$

Component amplitudes with power $\tilde{\eta}^{4k}$

- Planarity = single trace approx, ordered particles

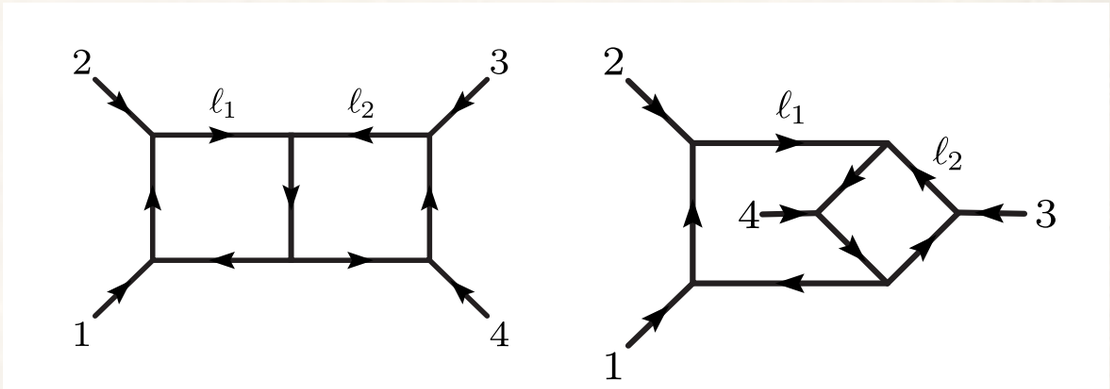
Simple amplitudes

- Comparison: Feynman diagrams vs unitary methods

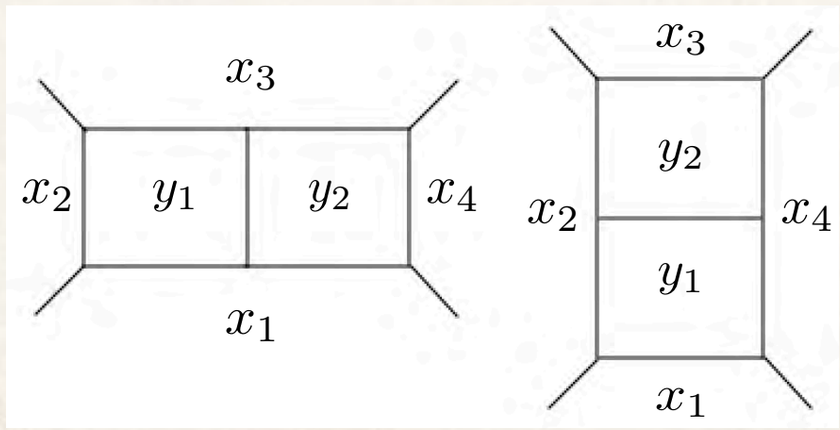
$gg \to gg$	Number of graphs	
	87 vs 1	
	~1000 vs 2	

Dual variables

- Generally, each diagram has its own variables
  - No global loop momenta
  - Each diagram: its own labels



- Planar limit: dual variables



$$k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3)$$
$$\ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3)$$

etc

Global variables

Integrand

- Using these variables: define a single function

$$\mathcal{M} = \int d^4y_1 \dots d^4y_L \, \mathcal{I}(x_i, y_j)$$

Integrand

- Ideal object to study: rational function, no divergencies

- Standard wisdom: $\mathcal{I} \sim \mathcal{I} + \text{Total derivative}$

- Planar N=4 SYM: unique function

Dual conformal invariance (DCI)

- Tree-level amplitudes + integrand in planar N=4 SYM:

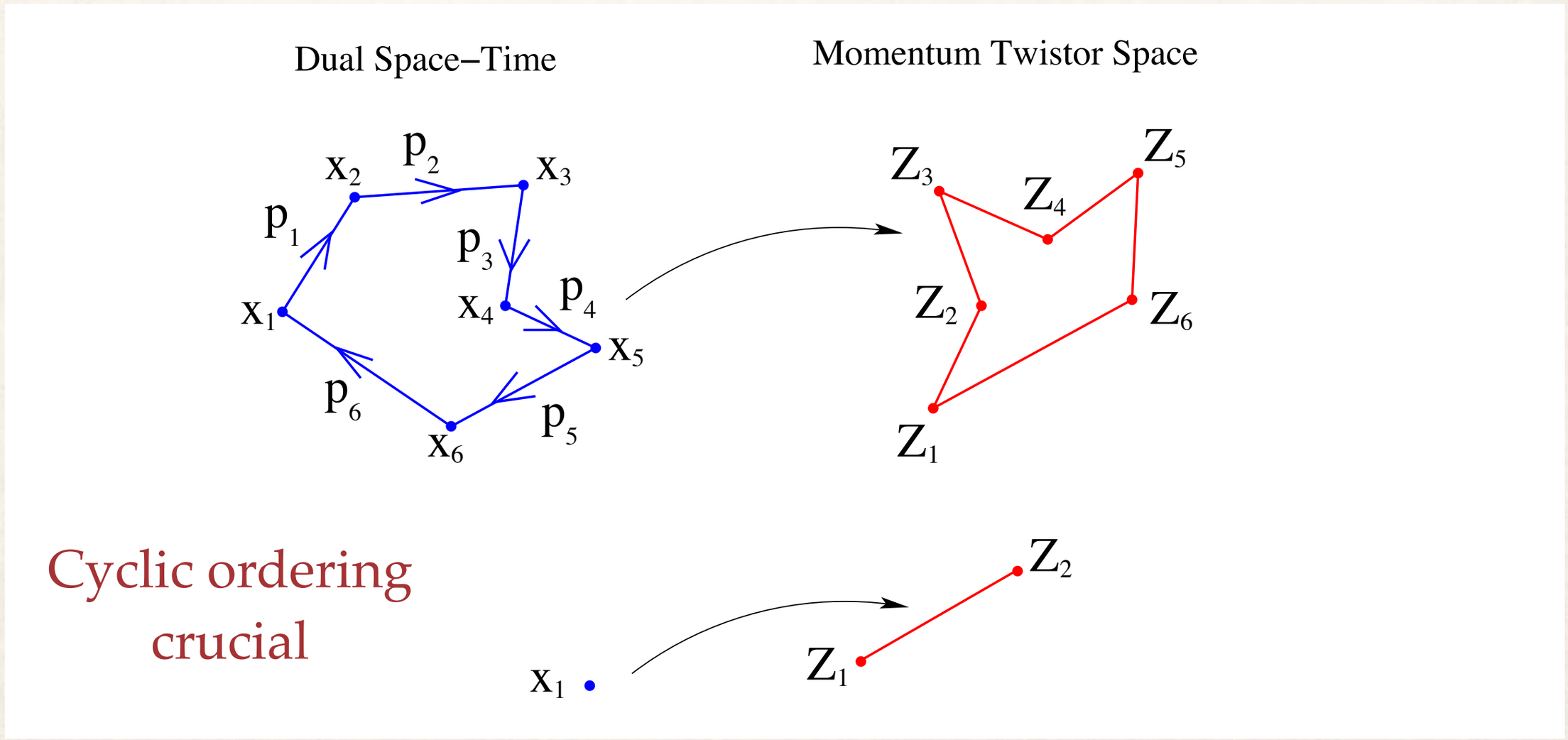
Dual conformal symmetry

- Dual variables $p_i = x_{i+1} - x_i$ $\sum_i x_i = 0$
- Conformal symmetry in the dual space
- Superconformal symmetry + Dual -> Yangian

Momentum twistors

- New variables: points in $\mathbb{P}^3$

$$Z = \begin{pmatrix} \lambda_a \\ x_{a\dot{a}}\tilde{\lambda}_{\dot{a}} \end{pmatrix}$$



Cyclic ordering crucial

Momentum twistors

- Dual conformal: SL(4) on momentum twistors
- Dual conformal invariants: $\langle 1234\rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d$

$$\langle 1234\rangle = \langle 12\rangle\langle 23\rangle\langle 34\rangle[23]$$

- Yangian invariants:

$$[12345] = \frac{(\eta_1\langle 2345\rangle + \cdots + \eta_5\langle 1234\rangle)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

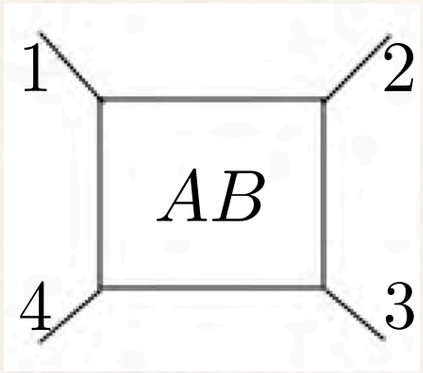
where $\widetilde{\eta}_a = \dfrac{\langle a-1\, a+1\rangle\eta_a + \langle a\, a-1\rangle\eta_{a+1} + \langle a+1\, a\rangle\eta_{a-1}}{\langle a\, a+1\rangle\langle a\, a-1\rangle}$

Momentum twistors

- Loop variable: pair of momentum twistors

$$\ell \leftrightarrow Z_A Z_B$$

- Example:



$$\frac{d^4\ell\, st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

$$\frac{\langle AB d^2 A\rangle\langle AB d^2 B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

where $\ell^2 = \dfrac{\langle AB41\rangle}{\langle AB\rangle\langle 41\rangle}$

$$\langle ij\rangle = \epsilon_{abcd} Z_i^a Z_j^b I^{cd}$$

Infinity twistor: breaking of DCI

DCI of trees and integrand

- In planar N=4 SYM

$$\mathcal{R}_{n,k}^{\ell-loop} = \frac{\mathcal{I}_{n,k}^{\ell-loop}}{\mathcal{M}_{n,k=2}^{tree}}$$

Yangian invariant (covariant)

Parke-Taylor amplitude

- Example of 6pt tree-level amplitude

$$\mathcal{M}_6 = [12345] + [34561] + [56123]$$

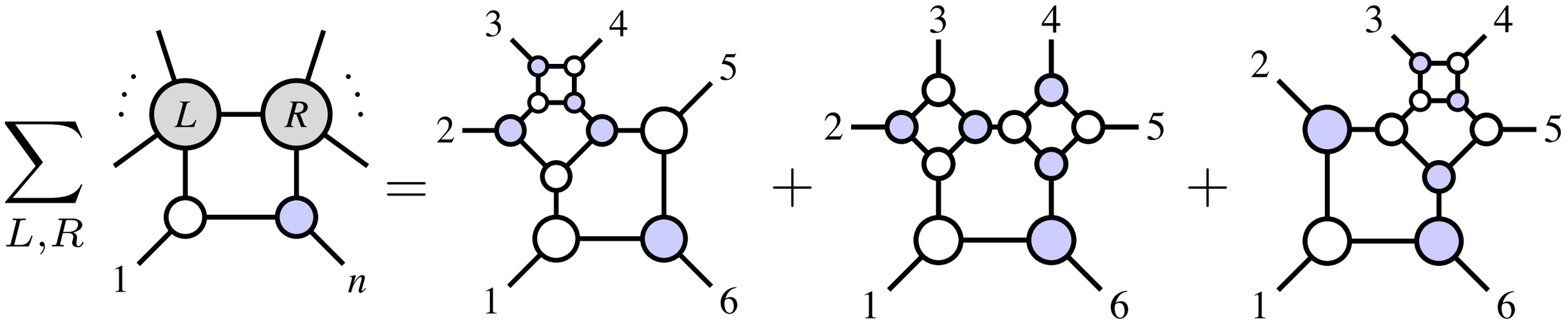
All terms include spurious poles in momentum space

Tension between hidden symmetry and locality

On-shell diagrams in momentum twistor space

- Momentum space -> momentum twistor space

$$k \to k-2 \qquad\qquad \delta(C\cdot Z)$$



$$= [12345] + [34561] + [56123]$$

Yangian invariance

Spurious poles

What is next?

- On-shell diagrams: building blocks for the amplitude

- Each corresponds to a volume form for the cell of the positive Grassmannian

- Complete geometric formulation: remove the knowledge of the recursion relations

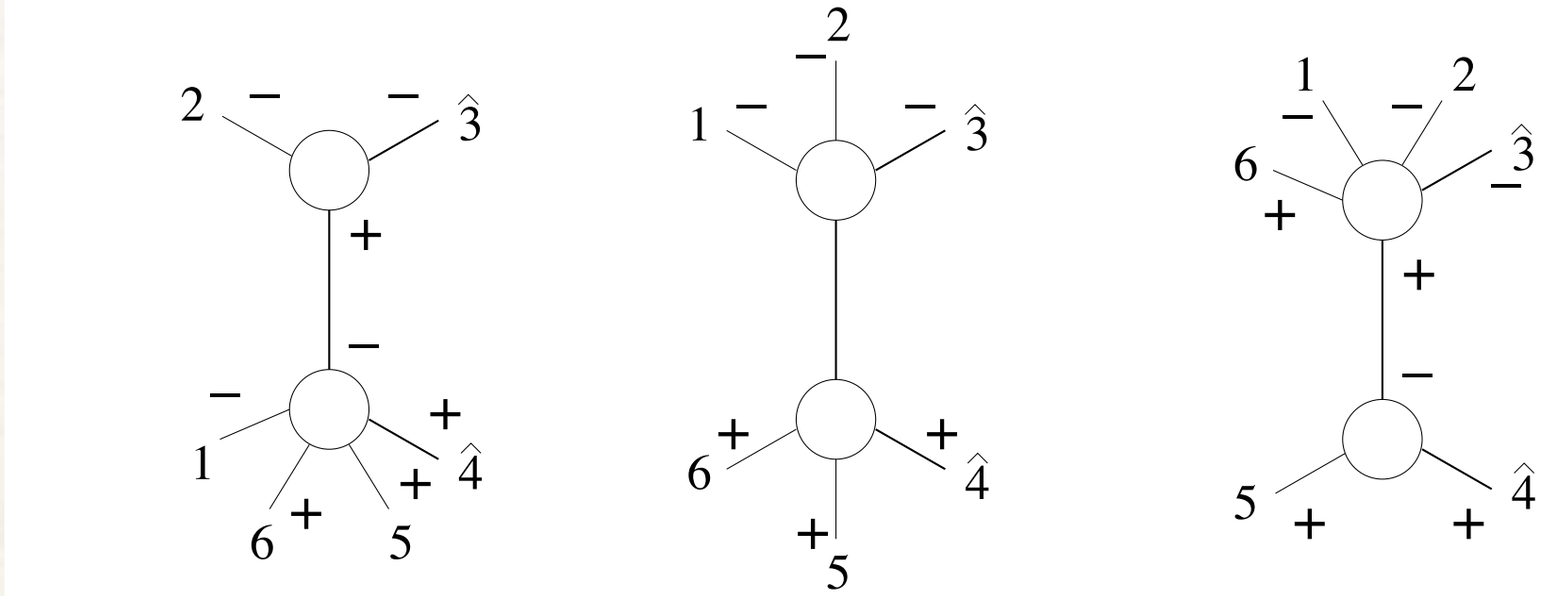
Prelude

Volume of polyhedron

(Hodges 2009)



- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

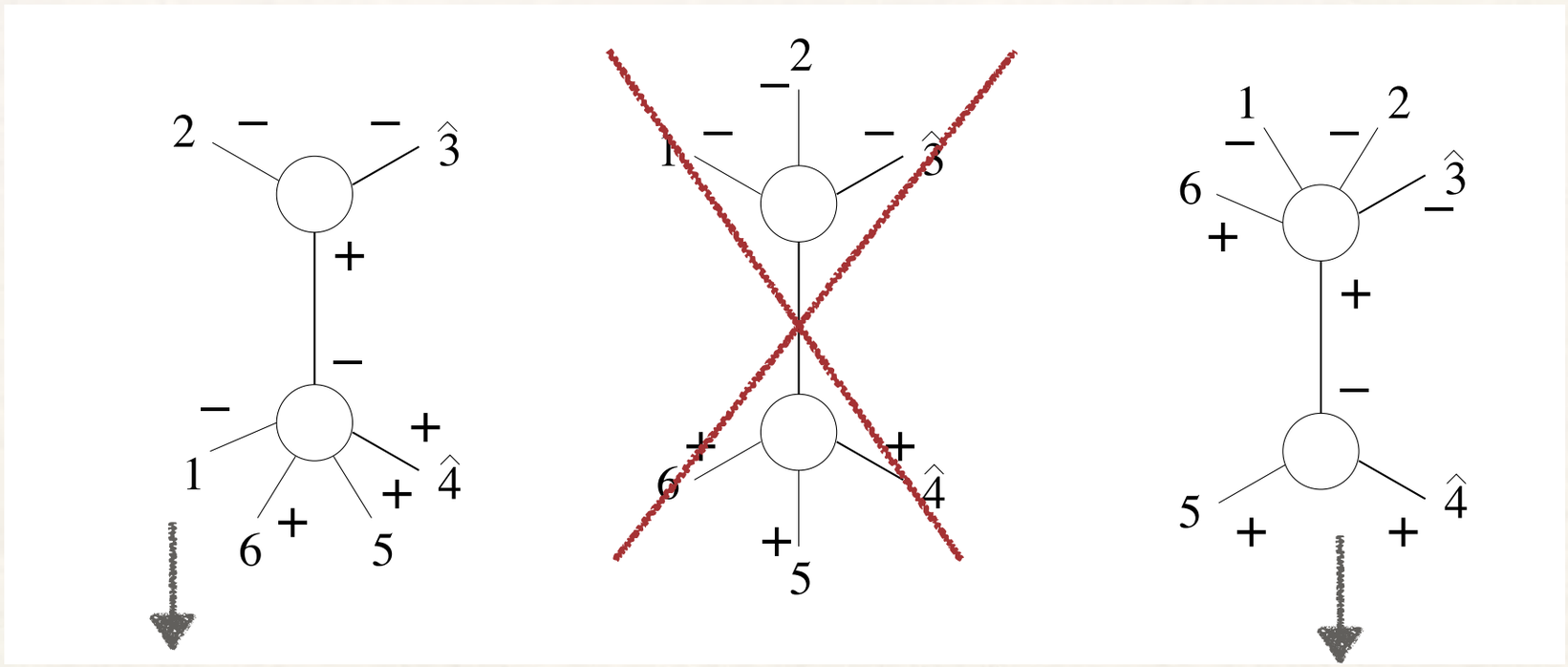


Volume of polyhedron

(Hodges 2009)



- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space



$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

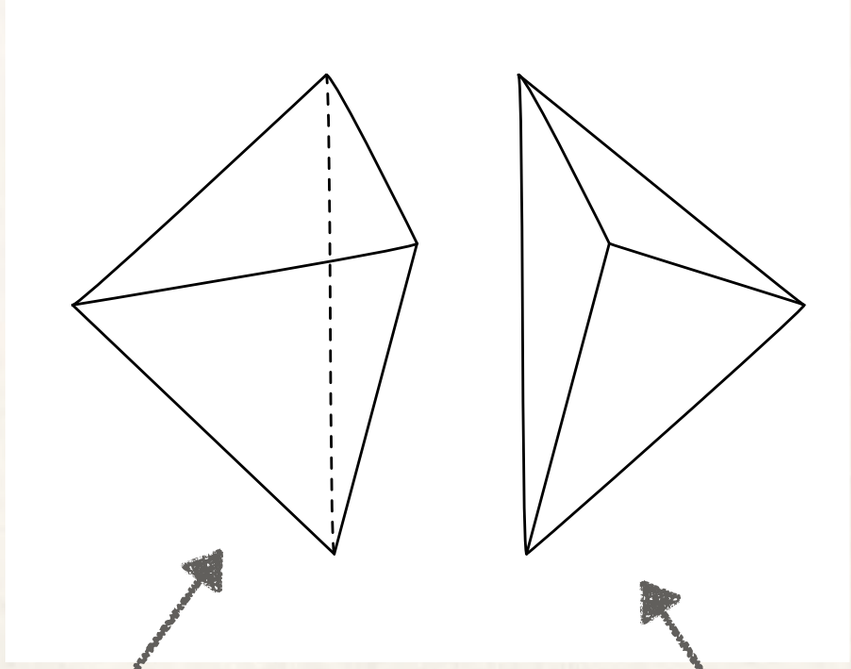
Volume of polyhedron

(Hodges 2009)



- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Volume of tetrahedron in momentum twistor space!



Each face labeled by $\langle abcd \rangle$

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

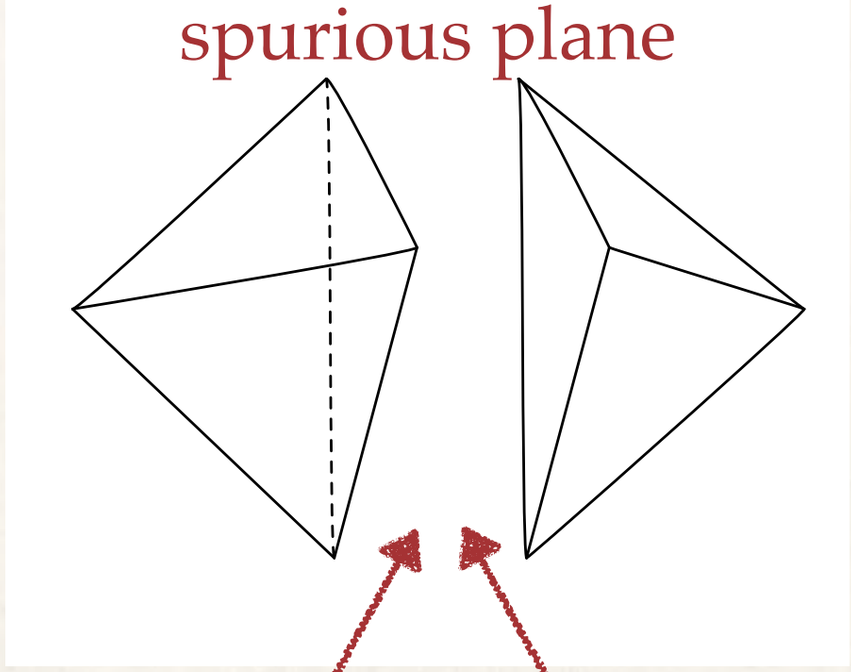
Volume of polyhedron

(Hodges 2009)



- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Volume of tetrahedron in momentum twistor space!



Each face labeled by $\langle abcd \rangle$

$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

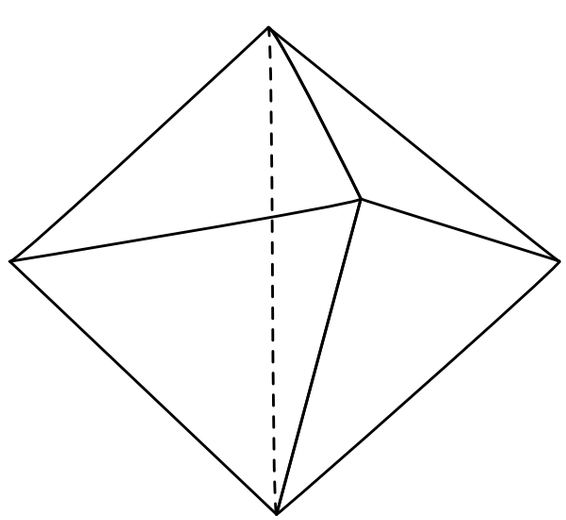
Volume of polyhedron

(Hodges 2009)



- Study tree-level scattering amplitude $A_6(1^-2^-3^-4^+5^+6^+)$
- BCFW recursion relations in momentum twistor space

Amplitude is a volume of polyhedron



Each face labeled by $\langle abcd \rangle$

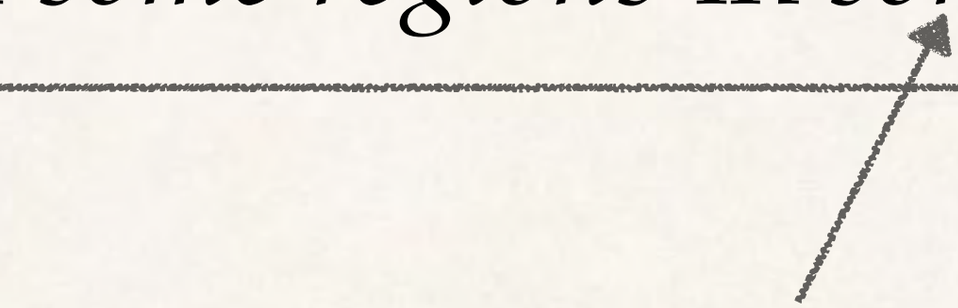
$$\frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 1235\rangle} \qquad \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 1236\rangle}$$

"Conjecture"

Amplitudes are volumes
of *some regions* in *some space*

"Conjecture"

Amplitudes are volumes
of *some regions* in *some space*



Must be related to
positive Grassmannian

Strategy

- Simple intuitive geometric ideas
- Use suitable mathematical language to describe them
- Generalize to more complicated (non-intuitive) cases

Thank you for attention!



Grassmannian Geometry of Scattering Amplitudes

LECTURE 4

Jaroslav Trnka

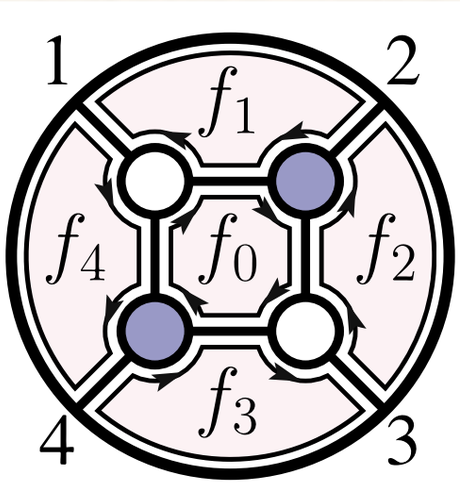
Center for Quantum Mathematics and Physics (QMAP), UC Davis

Amplitudes School, Edinburgh, July 2017

Review of previous lectures

Positive Grassmannian

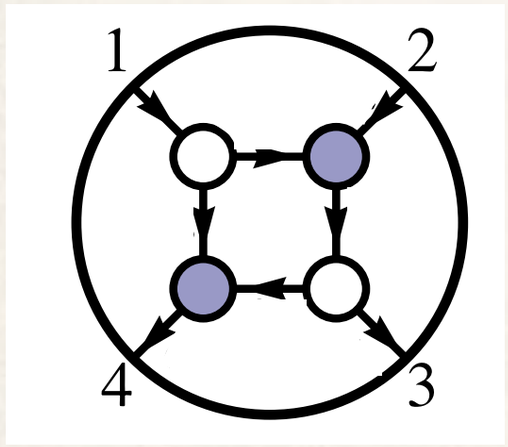
- Face variables



with the property

$$\prod_j f_j = -1$$

- Perfect orientation



Elements of $(k \times n)$ matrix

$$c_{ab} = -\sum_{\Gamma} \prod_j (-f_j)$$

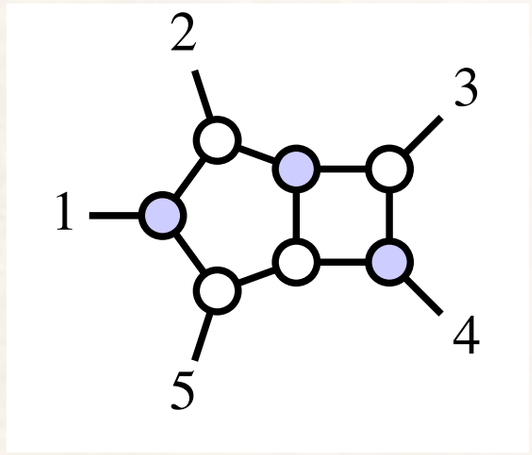
$$\overset{n}{k\begin{pmatrix} * & * & * & \dots & * \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * \end{pmatrix}}$$

Cell in positive Grassmannian $G_+(k,n)$

$$\overset{k}{k\begin{vmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{vmatrix}} \geq 0$$

On-shell diagrams for N=4 SYM

- For any on-shell diagram



only dependence on kinematics

$$R = \int \frac{df_1}{f_1}\frac{df_2}{f_2}\cdots\frac{df_m}{f_m}\,\delta(C\cdot Z)$$

$$\updownarrow$$

$$\mathcal{M}_1^{tree}\,\mathcal{M}_2^{tree}\,\ldots\mathcal{M}_m^{tree}$$

- Amplitude still given by BCFW: sum of on-shell diagrams

Unitarity

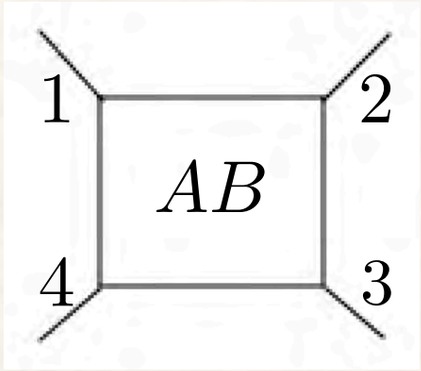
- New definition of amplitude wanted

Momentum twistors

- Dual conformal: SL(4) on momentum twistors
- Dual conformal invariants: $\langle 1234\rangle = \epsilon_{abcd} Z_1^a Z_2^b Z_3^c Z_4^d$

$$\langle 1234\rangle = \langle 12\rangle\langle 23\rangle\langle 34\rangle[23]$$

- Loop momenta: $\ell \leftrightarrow Z_A Z_B$



$$\frac{d^4\ell\, st}{\ell^2(\ell+k_1)^2(\ell+k_1+k_2)^2(\ell-k_4)^2}$$

$$\frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}$$

Yangian symmetry

- Superconformal + dual -> Yangian symmetry

(Drummond, Korchemsky, Sokatchev, Henn, 2008) (Drummond, Henn, Plefka, 2009)

- Example of Yangian invariants:

$$[12345] = \frac{(\eta_1\langle 2345\rangle + \cdots + \eta_5\langle 1234\rangle)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

- 6pt NMHV amplitude:

$$\mathcal{M}_6 = [12345] + [34561] + [56123]$$

Logarithmic form

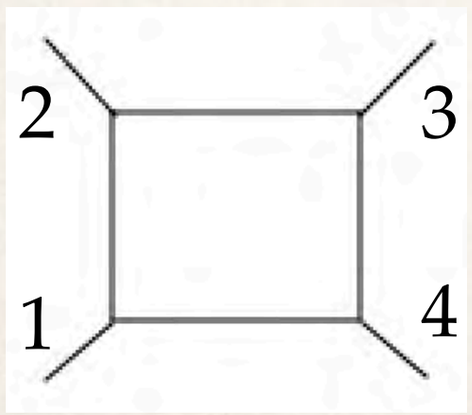
- Differential form with logarithmic singularities

- Simple examples:

$$\Omega \sim \frac{dx}{x} \quad \text{near} \quad x = 0$$

$$x > 0: \quad \text{Vol} = \frac{dx}{x} \qquad y > x > 0: \quad \text{Vol} = \frac{dx}{x}\frac{dy}{y-x}$$

- Integrals are also dlogs:

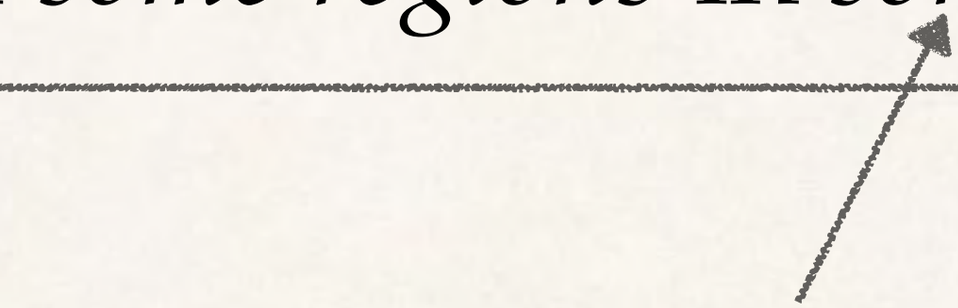


$$\text{dlog}\frac{\ell^2}{(\ell-\ell^*)^2}\text{dlog}\frac{(\ell+k_1^2)^2}{(\ell-\ell^*)^2}\text{dlog}\frac{(\ell+k_1+k_2)^2}{(\ell-\ell^*)^2}\text{dlog}\frac{(\ell-k_4)^2}{(\ell-\ell^*)^2}$$

$$\text{dlog}\frac{\langle AB12\rangle}{\langle AB13\rangle}\text{dlog}\frac{\langle AB23\rangle}{\langle AB13\rangle}\text{dlog}\frac{\langle AB34\rangle}{\langle AB13\rangle}\text{dlog}\frac{\langle AB41\rangle}{\langle AB13\rangle}$$

"Conjecture"

Amplitudes are volumes
of *some regions* in *some space*



Must be related to
positive Grassmannian

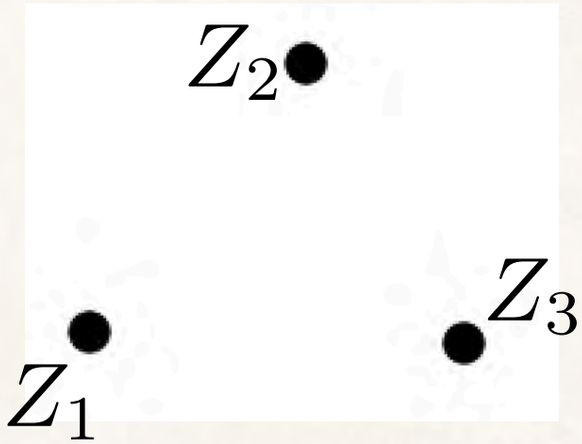
Strategy

- Simple intuitive geometric ideas
- Use suitable mathematical language to describe them
- Generalize to more complicated (non-intuitive) cases

Inside of the triangle

Inside of the triangle

- Let us consider three points in a projective plane

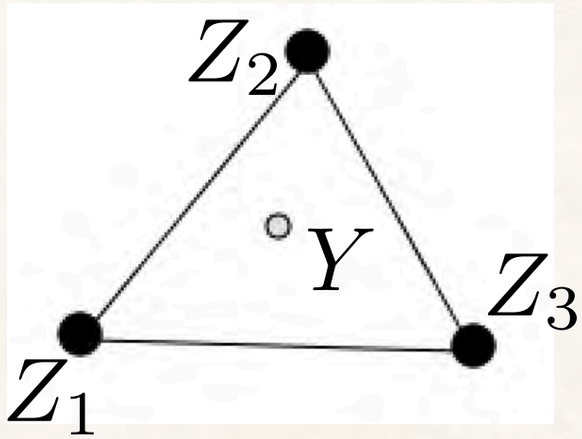


$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim tZ_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

Inside of the triangle

- Point inside the triangle



$$Z_j = \begin{pmatrix} * \\ * \\ * \end{pmatrix} \quad Z_j \sim tZ_j$$

We can also fix $Z_j = \begin{pmatrix} 1 \\ a_j \\ b_j \end{pmatrix}$

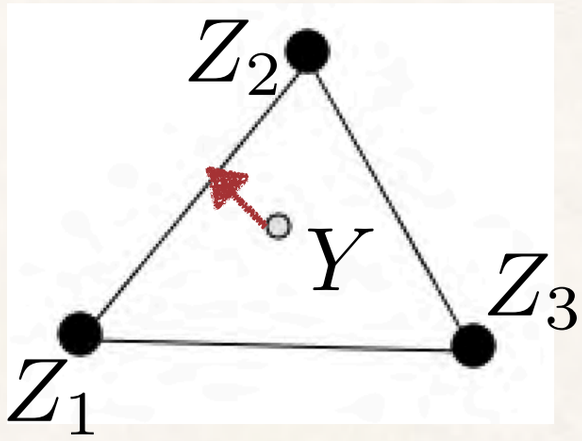
- Point inside the triangle

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 \qquad c_1, c_2, c_3 > 0$$

Projecte: one of c_j can be fixed to 1

Inside of the triangle

- Point inside the triangle



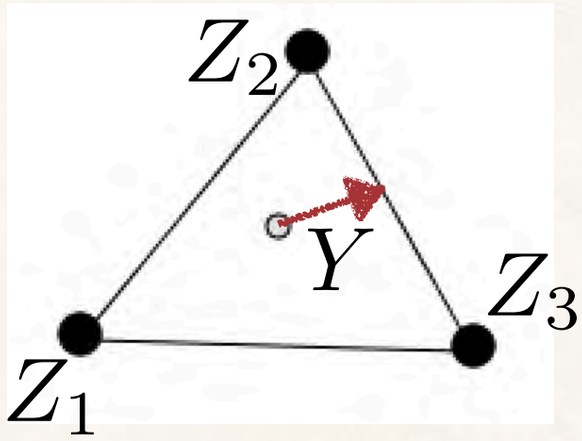
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_3 = 0$$

Inside of the triangle

- Point inside the triangle



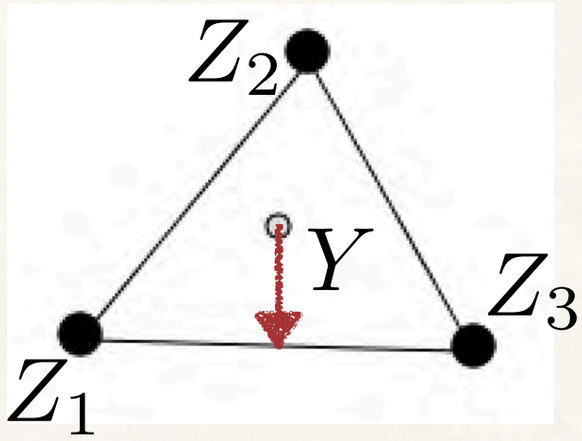
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_1 = 0$$

Inside of the triangle

- Point inside the triangle



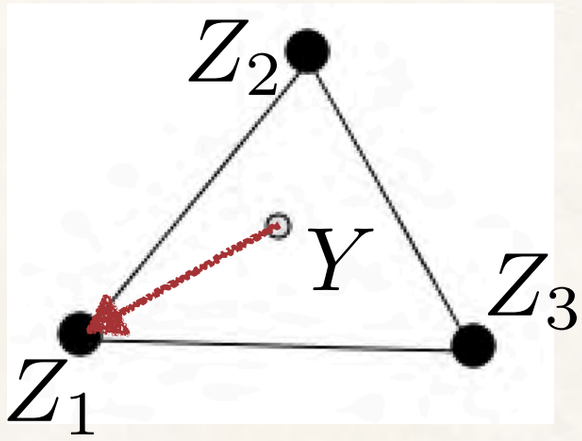
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = 0$$

Inside of the triangle

- Point inside the triangle



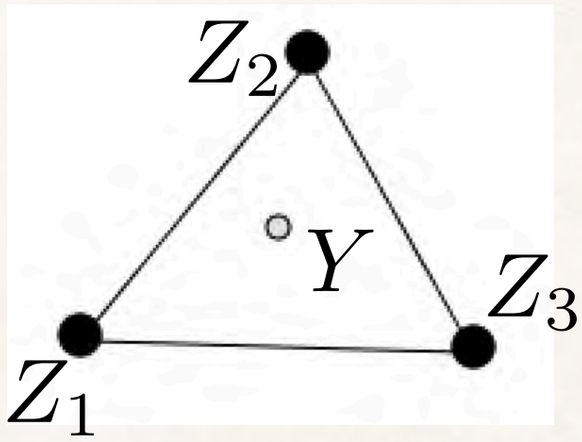
$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$

On the boundary

$$c_2 = c_3 = 0$$

Logarithmic form

- Point inside the triangle



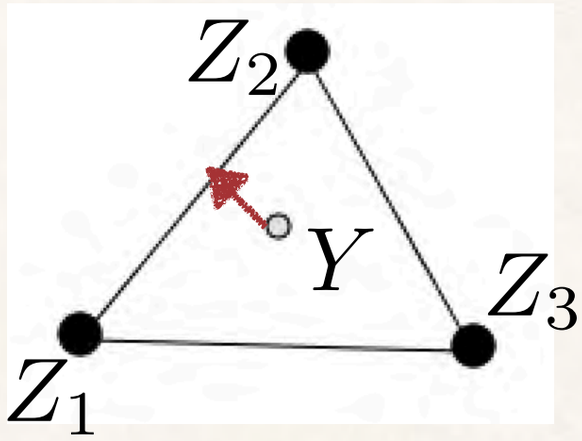
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_3 = 0$$

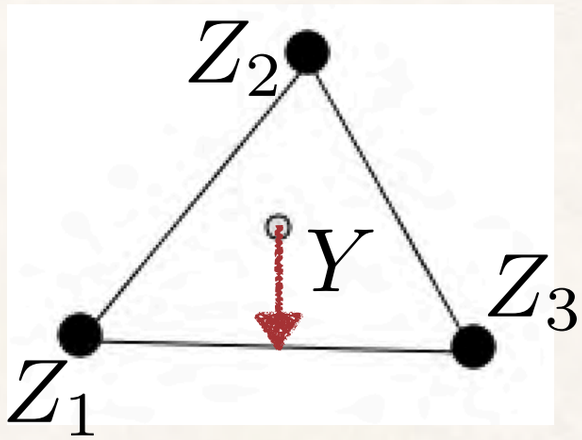
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_2}{c_2}$$

Logarithmic form

- Point inside the triangle



$$c_2 = 0$$

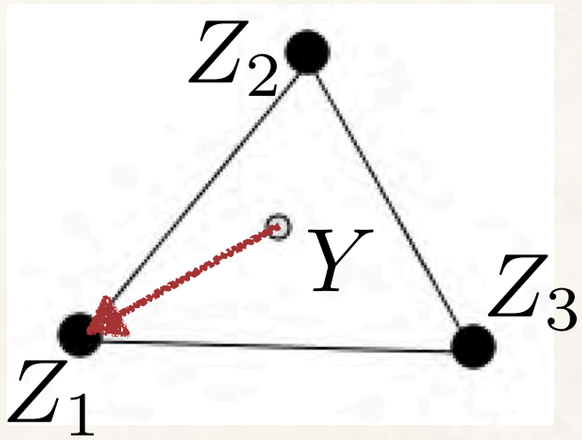
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_3}{c_3}$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

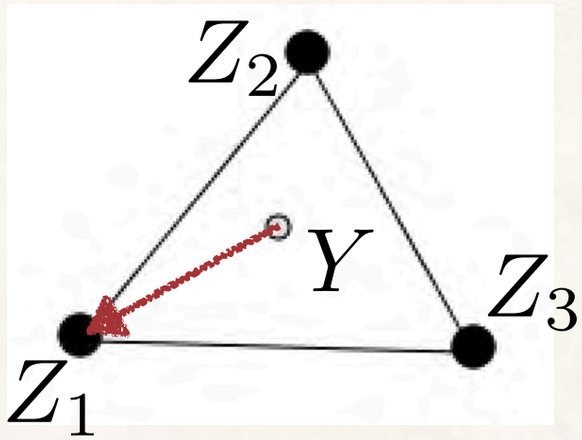
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_3}{c_3} \rightarrow 1$$

Logarithmic form

- Point inside the triangle



$$c_2 = c_3 = 0$$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \rightarrow \frac{dc_3}{c_3} \rightarrow 1$$

- Other boundaries can correspond to $c_2, c_3 \rightarrow \infty$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \qquad \langle X_1 X_2 X_3\rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

$$c_2 = \frac{\langle Y13\rangle}{\langle Y23\rangle} \quad c_3 = \frac{\langle Y12\rangle}{\langle Y23\rangle} \quad dc_2\, dc_3 = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y23\rangle^3}$$

Logarithmic form

- Form with logarithmic singularities on boundaries

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \qquad \langle X_1 X_2 X_3\rangle = \epsilon_{abc} X_1^a X_2^b X_3^c$$

$$d^2Y = dc_2\, dc_3\, Z_2 Z_3$$

- Solve for c_2, c_3 from $Y = Z_1 + c_2 Z_2 + c_3 Z_3$

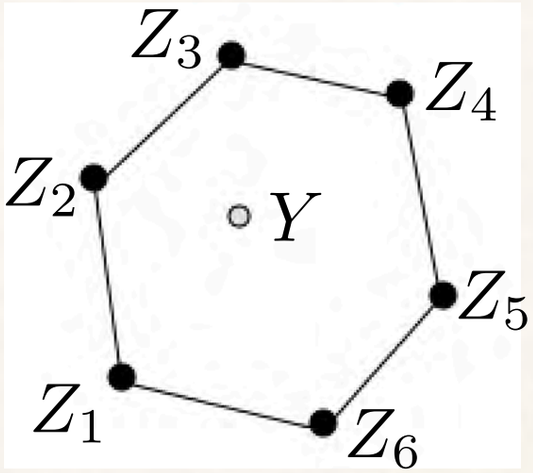
$$\Omega = \frac{\langle Y\, d^2Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

Projective in all variables

Polygon

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

$$\downarrow$$

$$c_j > 0$$

interior of the polygon

- Convex polygon: condition on points Z_i

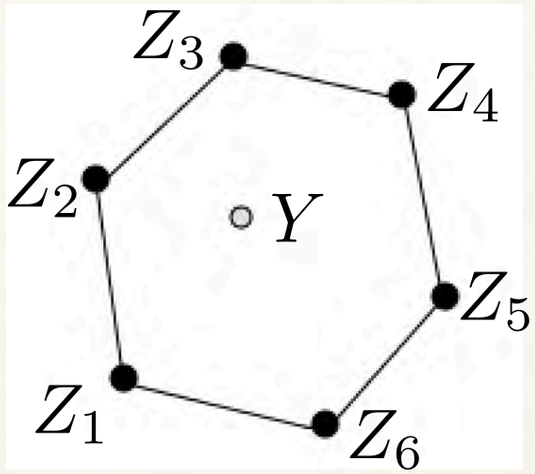
$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \dots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

All main minors positive

$$\begin{vmatrix} * & * & * \\ * & * & * \\ * & * & * \end{vmatrix} > 0$$

Point inside the polygon

- Consider a point inside a polygon in projective plane



$$Y = c_1 Z_1 + c_2 Z_2 + \dots c_n Z_n$$

Space of all points inside convex polygon

More formally:

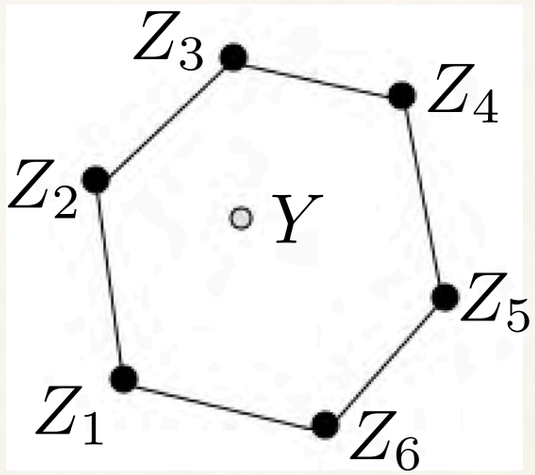
$$\boxed{Y = C \cdot Z}$$

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_n \end{pmatrix} \in G_+(1, n)$$

$$Z = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & \dots & Z_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \in M_+(3, n)$$

Logarithmic form

- Form with logarithmic singularities on boundaries



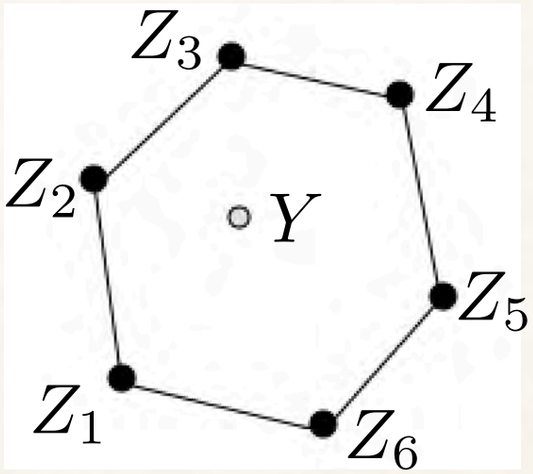
$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

First guess

$$\Omega = \frac{dc_1}{c_1} \frac{dc_2}{c_2} \ldots \frac{dc_n}{c_n}$$

Logarithmic form

- Form with logarithmic singularities on boundaries



$$Y = c_1 Z_1 + c_2 Z_2 + \ldots c_n Z_n$$

First guess

$$\Omega = \frac{dc_1}{c_1}\frac{dc_2}{c_2}\ldots\frac{dc_n}{c_n}$$

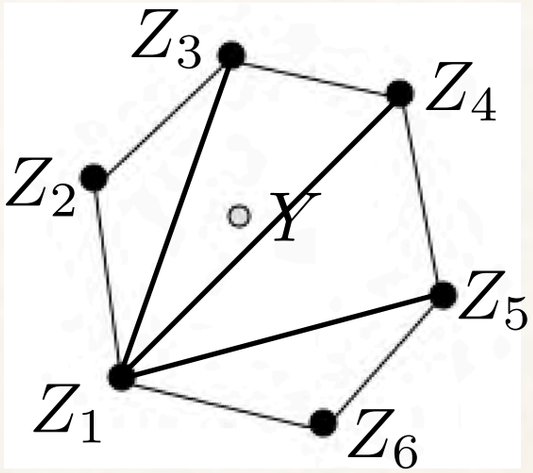
Space of all Y is only two-dimensional

- Two-form with n poles

$$\Omega \sim \frac{dc_1\, dc_2\, N(c_1, c_2)}{D(c_1, c_2)}$$

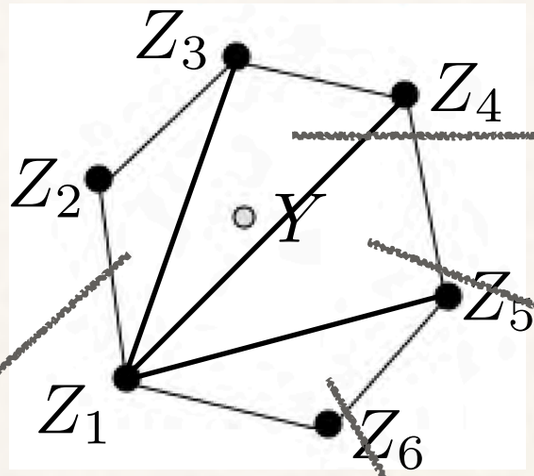
Logarithmic form

- Easiest way how to write the form is to triangulate



Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{dc_3}{c_3}\frac{dc_4}{c_4} \quad c_3, c_4 \geq 0$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{dc_4}{c_4}\frac{dc_5}{c_5} \quad c_4, c_5 \geq 0$$

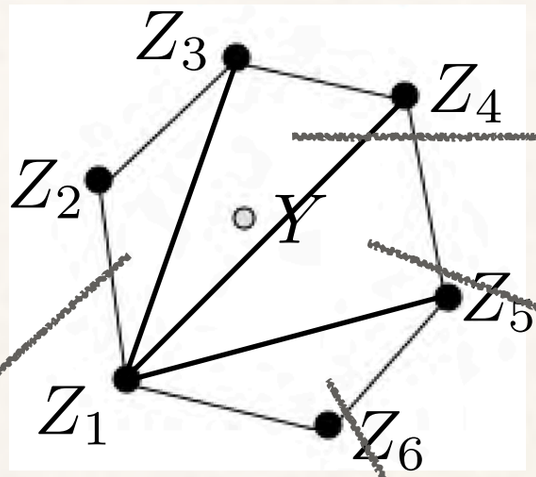
$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{dc_2}{c_2}\frac{dc_3}{c_3} \quad c_2, c_3 \geq 0$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{dc_5}{c_5}\frac{dc_6}{c_6} \quad c_5, c_6 \geq 0$$

How to sum them?

Logarithmic form

- Easiest way how to write the form is to triangulate



$$Y = Z_1 + c_3 Z_3 + c_4 Z_4$$
$$\Omega_2 = \frac{\langle Y\, d^2Y\rangle \langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle}$$

$$Y = Z_1 + c_4 Z_4 + c_5 Z_5$$
$$\Omega_3 = \frac{\langle Y\, d^2Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3$$
$$\Omega_1 = \frac{\langle Y\, d^2Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

$$Y = Z_1 + c_5 Z_5 + c_6 Z_6$$
$$\Omega_4 = \frac{\langle Y\, d^2Y\rangle \langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

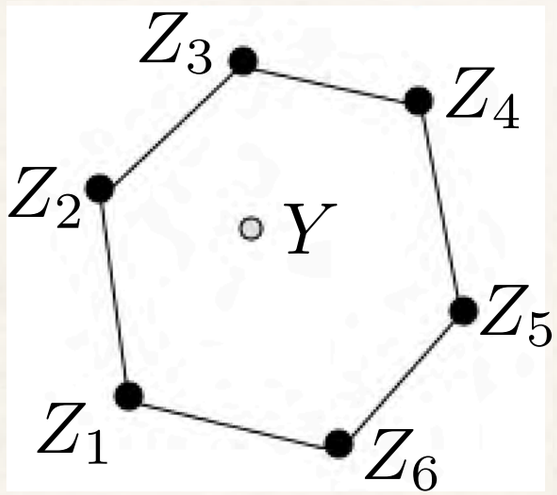
Write in projective form

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$

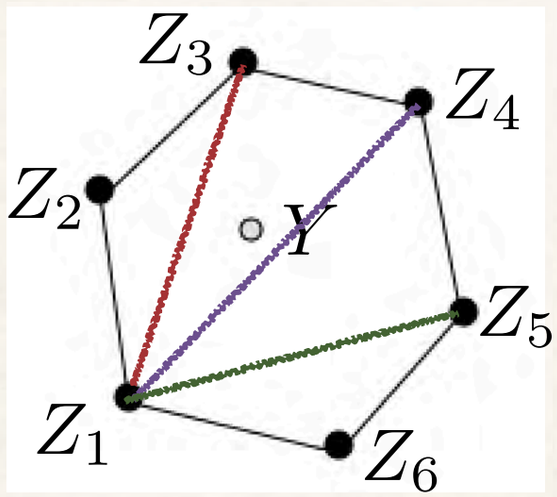


Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle\langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$



Spurious poles

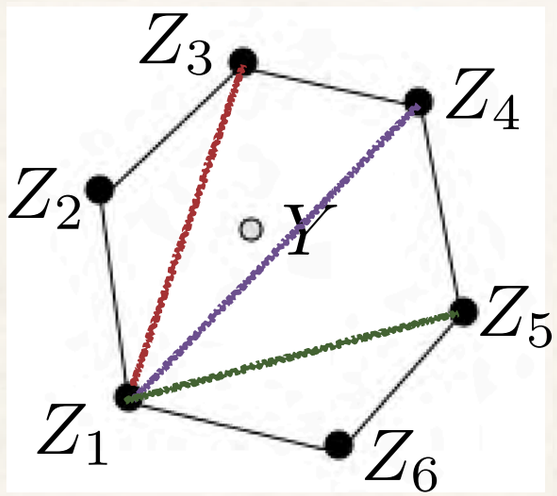
Cancel in the sum

Logarithmic form

- Now it makes sense to sum them

$$\Omega = \frac{\langle Y\, d^2Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y\, d^2Y\rangle \langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y\, d^2Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y\, d^2Y\rangle \langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

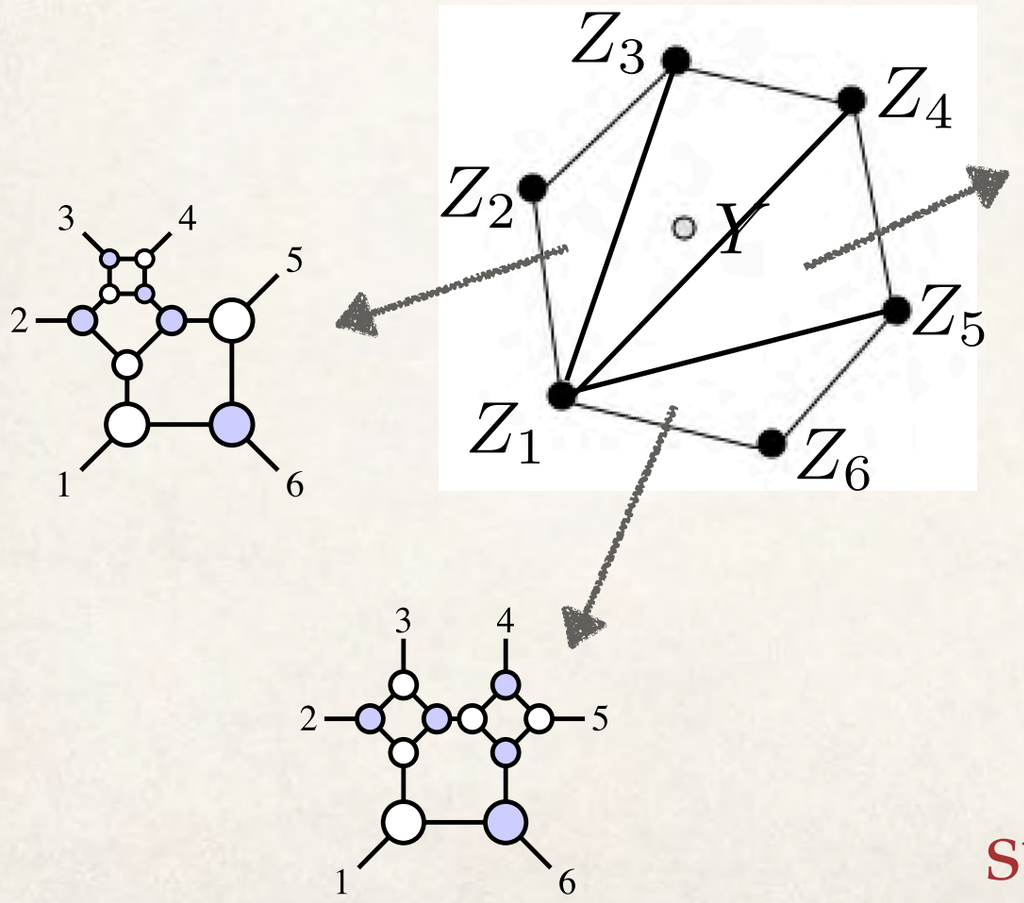
- Boundaries of the polygon are $\langle Y\, i\, i+1\rangle = 0$



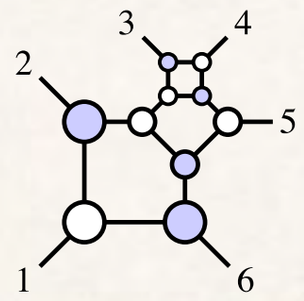
$$\Omega = \frac{\langle Y\, d^2Y\rangle\, \mathcal{N}(Y, Z_j)}{\langle Y12\rangle\langle Y23\rangle\langle Y34\rangle\langle Y45\rangle\langle Y56\rangle\langle Y61\rangle}$$

Similarities with on-shell diagrams

- Notice some similarities with recursion relations and on-shell diagrams



$$C = \begin{pmatrix} 1 & 0 & 0 & c_4 & c_5 & 0 \end{pmatrix} \in G_+(1,6)$$



$$\Omega = \frac{dc_4}{c_4}\frac{dc_5}{c_5}$$ looks similar

$$Y = C \cdot Z \qquad \delta(C \cdot Z)$$

$$\Omega = \frac{\langle Y\, d^2Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle}$$

On-shell diagrams:

supersymmetric, no Y

From Y to supersymmetry

- Let us take the form for the triangle

$$\Omega = \frac{\langle Y\, d^2 Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

- Rewrite external Z:

$$Z_j = \begin{pmatrix} z_j^{(1)} \\ z_j^{(2)} \\ (\phi\cdot\eta_j) \end{pmatrix} \qquad \begin{aligned} z_j &\in \mathbb{P}^2 \quad \text{bosonic} \\ \eta_j^A &\quad \text{ferimionic} \\ \phi^A &\quad \text{auxiliary} \end{aligned} \qquad A = 1,2$$

- Also define

$$Y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Consider $\int \Omega\, \delta(Y - Y_0)$

Effectively set: $Y = Y_0$

From Y to supersymmetry

Calculate: $\langle 123\rangle \to \begin{vmatrix} z_1^{(1)} & z_2^{(1)} & z_3^{(1)} \\ z_1^{(2)} & z_2^{(2)} & z_3^{(2)} \\ (\phi\cdot\eta_1) & (\phi\cdot\eta_2) & (\phi\cdot\eta_3) \end{vmatrix} = \begin{array}{c} \langle 12\rangle(\phi\cdot\eta_3) \\ +\langle 23\rangle(\phi\cdot\eta_1) \\ +\langle 31\rangle(\phi\cdot\eta_2) \end{array}$

$$\langle Y12\rangle \to \begin{vmatrix} 0 & z_1^{(1)} & z_2^{(1)} \\ 0 & z_1^{(2)} & z_2^{(2)} \\ 1 & (\phi\cdot\eta_1) & (\phi\cdot\eta_2) \end{vmatrix} = \langle 12\rangle$$

$$\langle Y23\rangle \to \begin{vmatrix} 0 & z_2^{(1)} & z_3^{(1)} \\ 0 & z_2^{(2)} & z_3^{(2)} \\ 1 & (\phi\cdot\eta_2) & (\phi\cdot\eta_3) \end{vmatrix} = \langle 23\rangle$$

$$\langle Y31\rangle \to \begin{vmatrix} 0 & z_3^{(1)} & z_1^{(1)} \\ 0 & z_3^{(2)} & z_1^{(2)} \\ 1 & (\phi\cdot\eta_3) & (\phi\cdot\eta_1) \end{vmatrix} = \langle 31\rangle$$

Old variables

$$Z_j = \begin{pmatrix} z_j^{(1)} \\ z_j^{(2)} \\ (\phi\cdot\eta_j) \end{pmatrix}$$

where new invariants are

$$\langle ij\rangle = \epsilon_{ab} z_i^a z_j^b$$

From Y to supersymmetry

- We plug them into the form for triangle:

$$\frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \to \frac{(\langle 12\rangle(\phi\cdot\eta_3)+\langle 23\rangle(\phi\cdot\eta_1)+\langle 31\rangle(\phi\cdot\eta_2))^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

- Final step: integrate over ϕ:

$$\int d^2\phi \int \Omega\,\delta(Y-Y_0) = \boxed{\frac{(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

From Y to supersymmetry

- We plug them into the form for triangle:

$$\frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \to \frac{(\langle 12\rangle(\phi\cdot\eta_3)+\langle 23\rangle(\phi\cdot\eta_1)+\langle 31\rangle(\phi\cdot\eta_2))^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

- Final step: integrate over ϕ:

$$\int d^2\phi \int \Omega\,\delta(Y-Y_0) = \boxed{\frac{(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

This remind the Yangian invariant:

$$\frac{(\langle 1234\rangle\eta_5+\langle 2345\rangle\eta_1+\langle 3451\rangle\eta_2+\langle 4512\rangle\eta_3+\langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

From Y to supersymmetry

- We plug them into the form for triangle:

$$\frac{\langle Y\, d^2Y\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \to \frac{(\langle 12\rangle(\phi\cdot\eta_3)+\langle 23\rangle(\phi\cdot\eta_1)+\langle 31\rangle(\phi\cdot\eta_2))^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

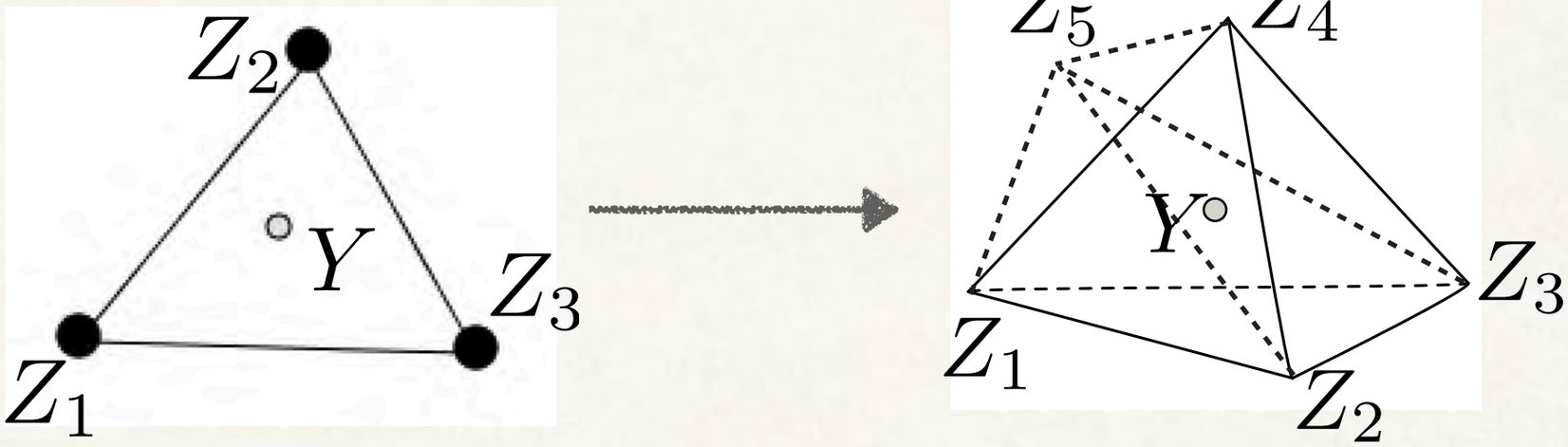
- Final step: integrate over ϕ:

$$\int d^2\phi \int \Omega\,\delta(Y-Y_0) = \boxed{\frac{(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}}$$

Yangian: $SL(4|4) \to SL(2|2)$

Repeating exercise

- We can repeat the exercise:



Point inside triangle ⟶ Point inside simplex

$$\frac{(\langle 12\rangle\eta_3+\langle 23\rangle\eta_1+\langle 31\rangle\eta_2)^2}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \longrightarrow \frac{(\langle 1234\rangle\eta_5+\langle 2345\rangle\eta_1+\langle 3451\rangle\eta_2+\langle 4512\rangle\eta_3+\langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

Amplituhedron

Polygon

- Inside of polygon:

$$\boxed{Y = C \cdot Z} \qquad C = \begin{pmatrix} * & * & \dots & * \end{pmatrix} \in G_+(1,n)$$

$$Z = \begin{pmatrix} Z_1 & Z_2 & \dots & Z_n \end{pmatrix} \in M_+(3,n)$$

- Form with logarithmic singularities on boundaries Ω

- Kinematics

$$Z_j = \begin{pmatrix} z_j \\ (\phi \cdot \eta_j) \end{pmatrix} \qquad Y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Final result

$$\boxed{A = \int d^2\phi \int \Omega\, \delta(Y - Y_0)}$$

Tree-level amplitudes

- Inside of polygon:

$$\boxed{Y = C \cdot Z}$$

$$C = \begin{pmatrix} * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \in G_+(k,n)$$

$$Z = \begin{pmatrix} Z_1 & Z_2 & \dots & Z_n \end{pmatrix} \in M_+(k+4,n)$$

- Form with logarithmic singularities on boundaries Ω

- Kinematics

$$Z_j = \begin{pmatrix} z_j \\ (\phi_1 \cdot \eta_j) \\ \vdots \\ (\phi_k \cdot \eta_j) \end{pmatrix} \qquad Y_0 = \begin{pmatrix} \mathbf{O}_{4\times 4} \\ \mathbf{1}_{k\times k} \end{pmatrix}$$

Final result

$$\boxed{A = \int d^4\phi_1 \dots d^4\phi_k \int \Omega\, \delta(Y - Y_0)}$$

Tree-level amplitude in N=4 SYM $\mathcal{A}_{n,k}$

Loop amplitudes

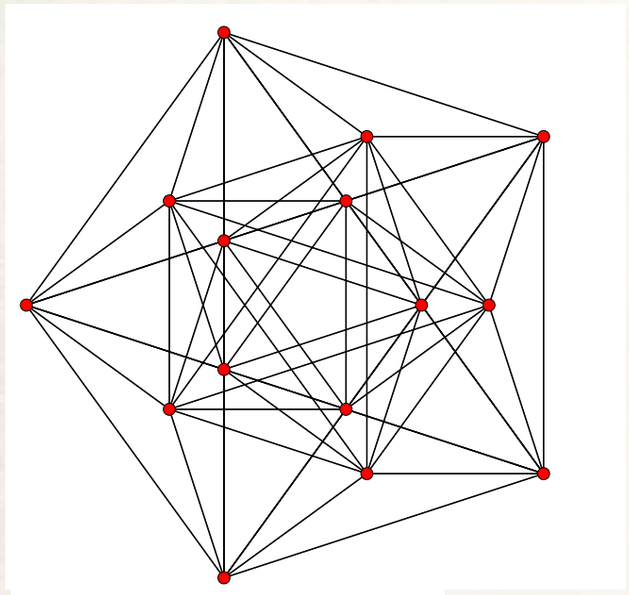
- Generalization to loops

$$\mathcal{Y} = \mathcal{C} \cdot Z$$

$$C = \begin{pmatrix} * & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * \end{pmatrix} \longrightarrow \mathcal{C} = \begin{pmatrix} C \\ D_1 \\ \vdots \\ D_\ell \end{pmatrix} \text{such that} \begin{pmatrix} C \\ D_{i_1} \\ \vdots \\ D_{i_m} \end{pmatrix} \in G_+(k+2m, n) \text{ for all } 0 \leq m \leq \ell$$

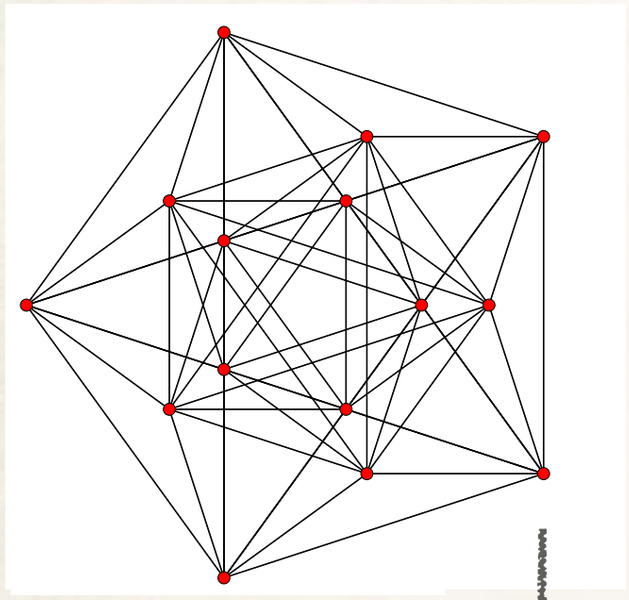
Integrand of amplitudes in planar N=4 SYM $\mathcal{I}_{n,k}^{\ell-loop}$

Triangulation



space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

Triangulation



space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

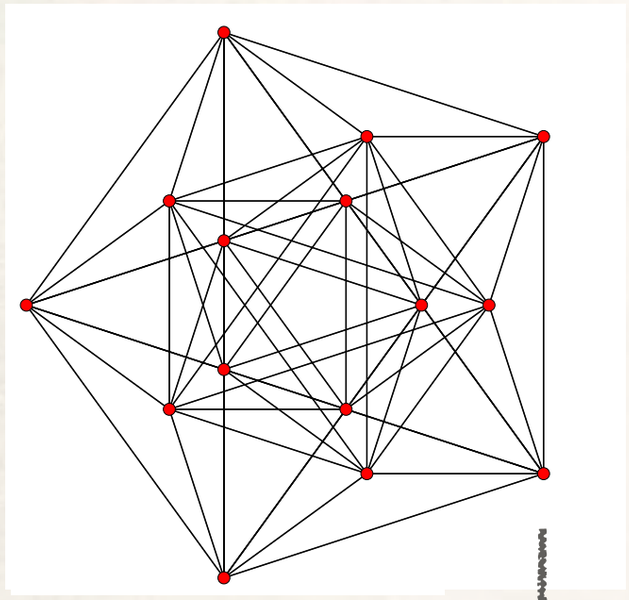
triangulate in terms of "simplices" $\Omega_0 \sim \frac{dx}{x}$ for each →

Set of regions:

- cover the whole space
- each region specified by $f_j \in (0, \infty)$

→ sum them

Triangulation



space specified by a set of inequalities → Ω logarithmic singularities → $\mathcal{M}_{n,k}^{\ell-loop}$

triangulate in terms of "simplices"

$\Omega_0 \sim \frac{dx}{x}$ for each → Set of regions:

- cover the whole space
- each region specified by $f_j \in (0, \infty)$

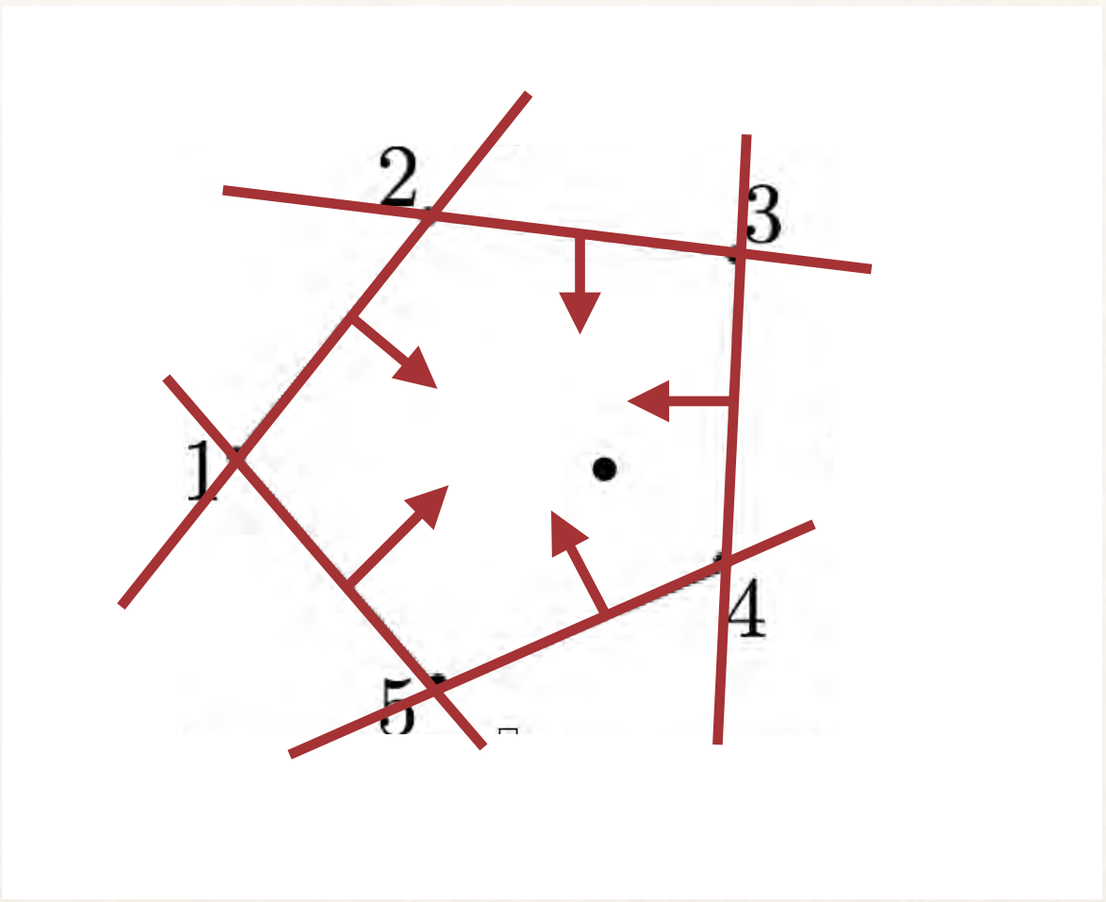
→ sum them

One explicit example:

4pt scattering to all loops

Different description

- There is a second way how to describe polygon



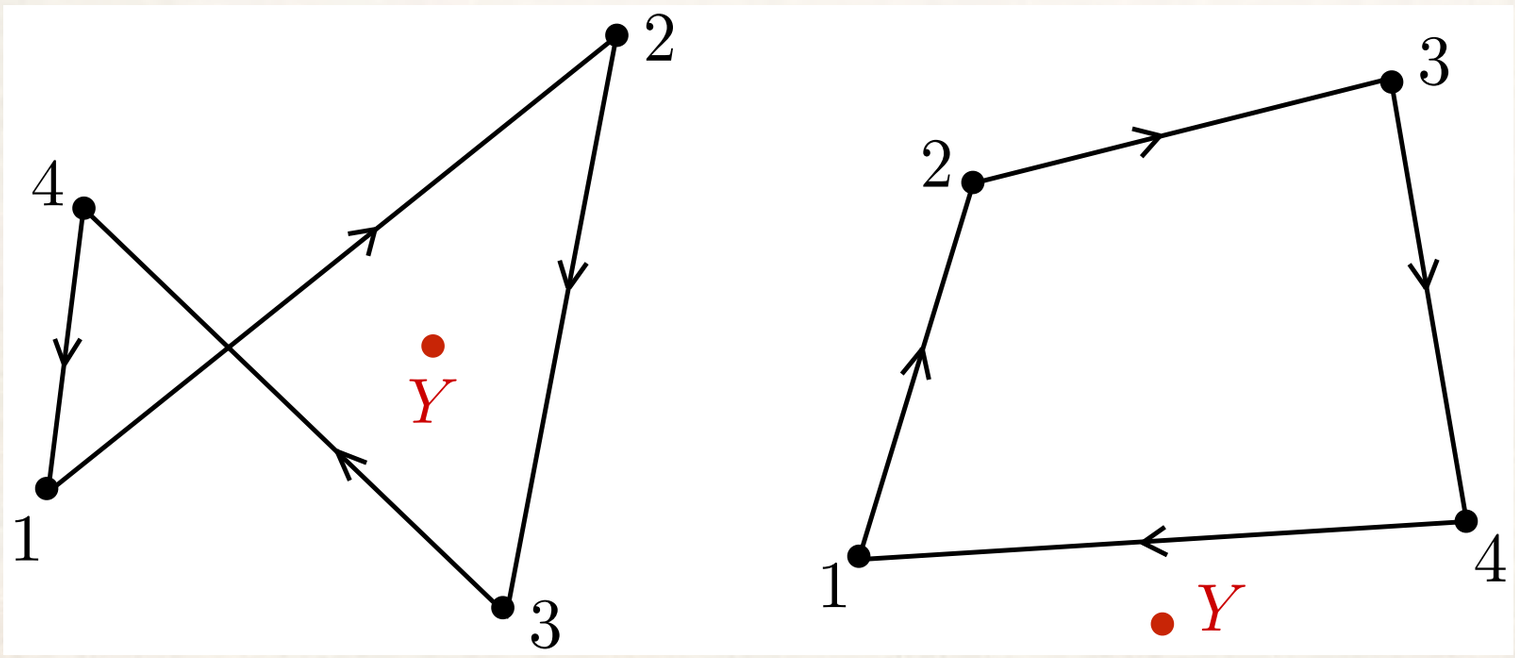
Using “faces” of the polygon

- This description desired for several reasons

Unwinding the Amplituhedron

(Arkani-Hamed, Thomas, Trnka 2017)

- Does not generalize: being on the right side of all faces not enough to specify the space
- Projection to 2d picture:

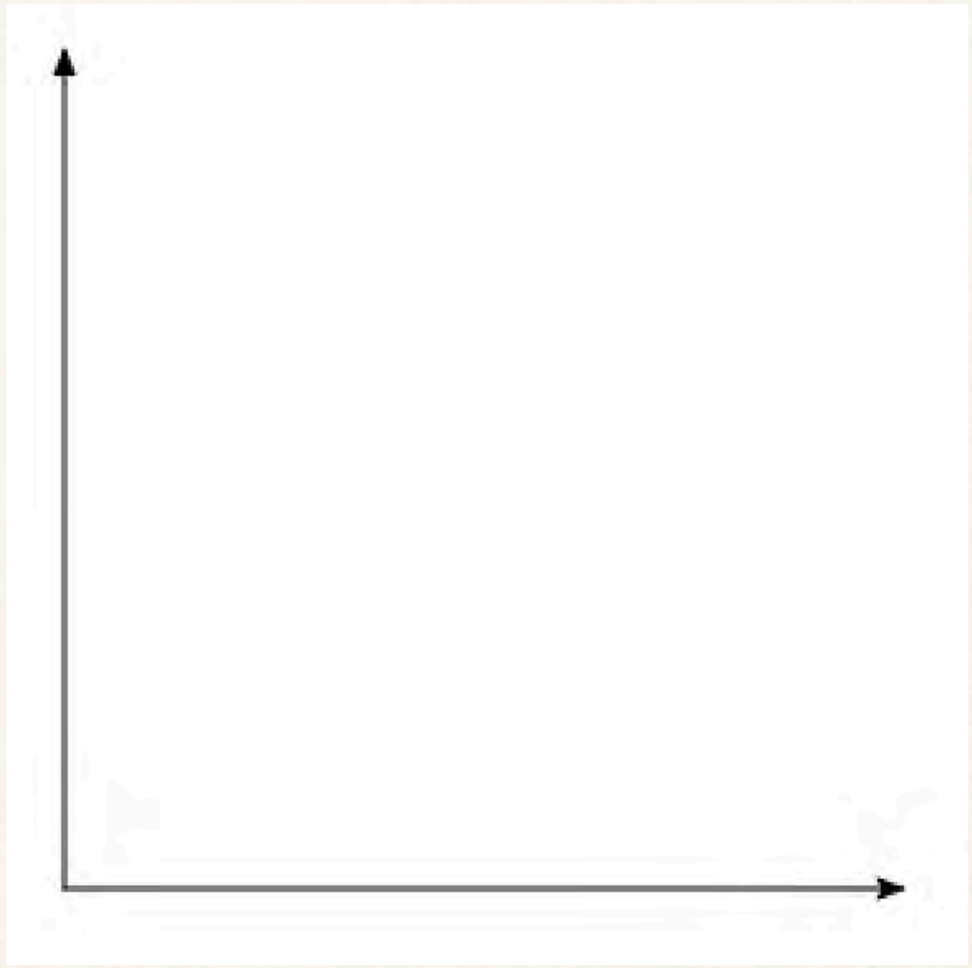


Amplituhedron: maximal "winding"

High school problem $gg \to gg$

High school problem $gg \to gg$

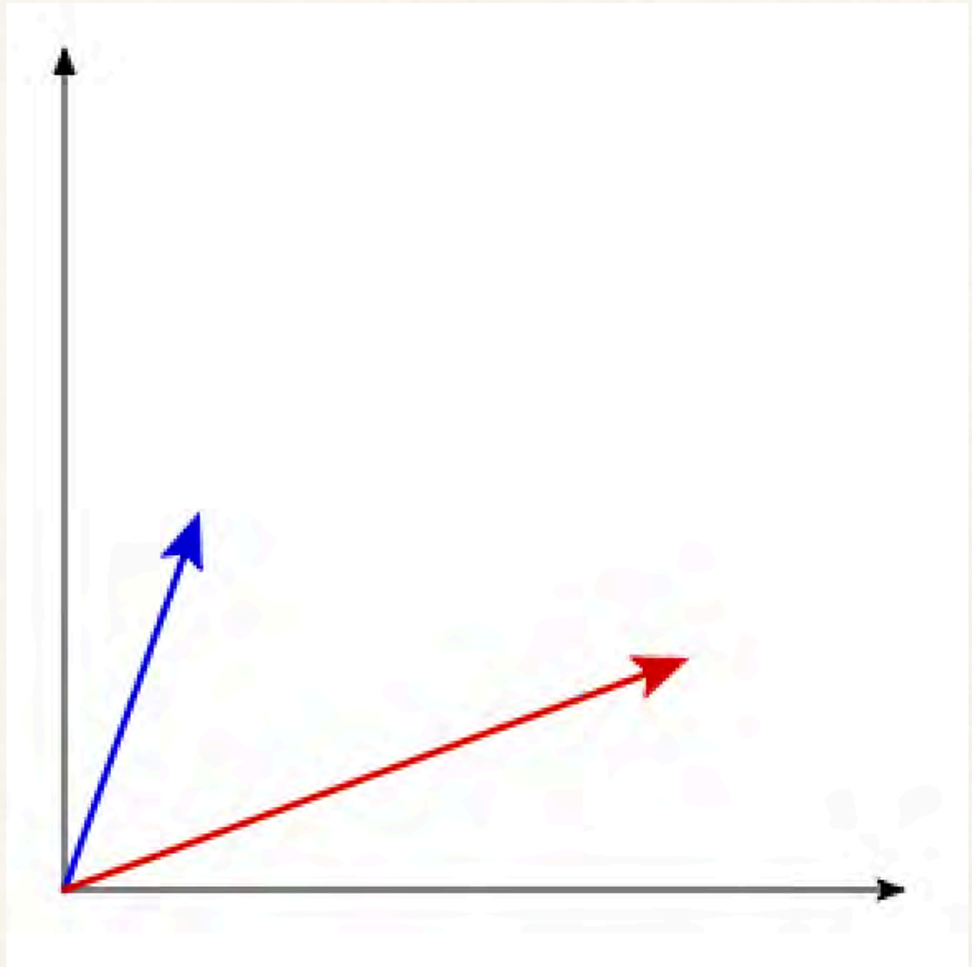
- Positive quadrant



High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



$$\text{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

High school problem

- Change of variables

where $\ell^* = \dfrac{[12]}{[24]}\lambda_1\widetilde{\lambda}_4$

$$x_1 = \frac{\ell^2}{(\ell-\ell^*)^2} = \frac{\langle AB41\rangle}{\langle AB13\rangle} \qquad z_1 = \frac{(\ell+k_1+k_2)^2}{(\ell-\ell^*)^2} = \frac{\langle AB23\rangle}{\langle AB13\rangle}$$

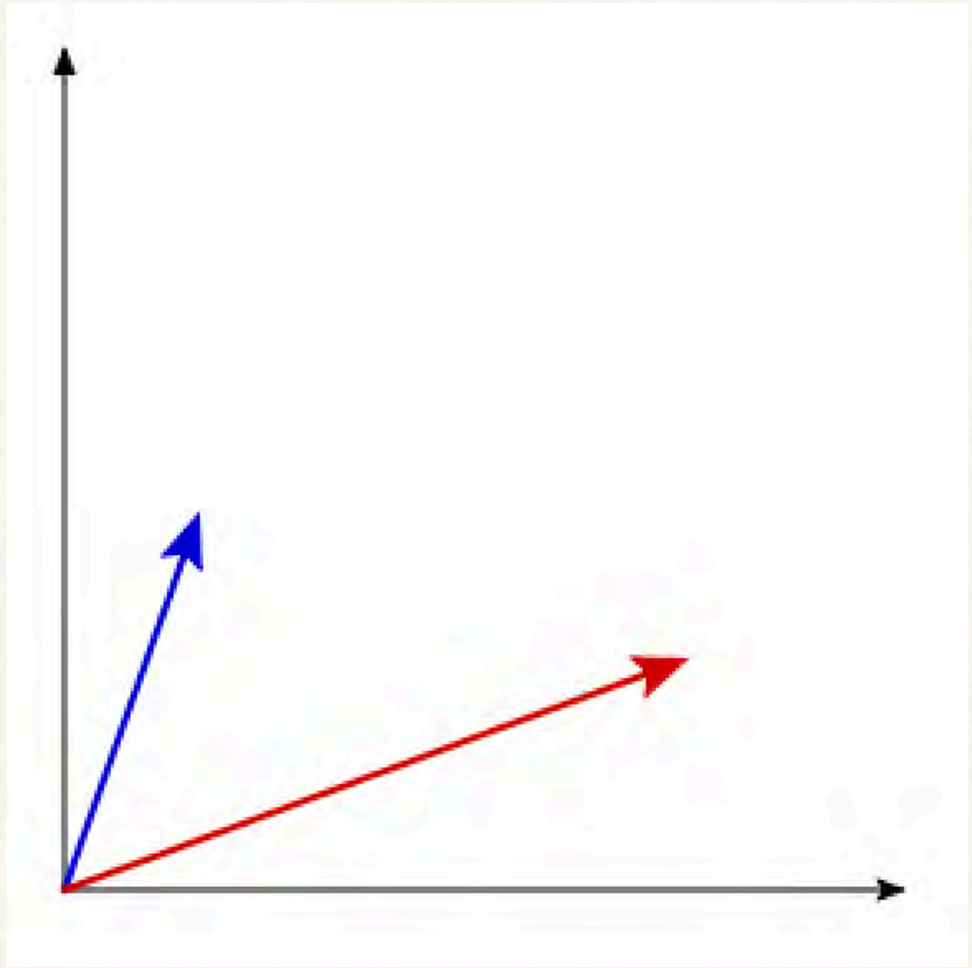
$$y_1 = \frac{(\ell+k_1)^2}{(\ell-\ell^*)^2} = \frac{\langle AB12\rangle}{\langle AB13\rangle} \qquad w_1 = \frac{(\ell-k_4)^2}{(\ell-\ell^*)^2} = \frac{\langle AB34\rangle}{\langle AB13\rangle}$$

Exactly the same expressions
true for all loops

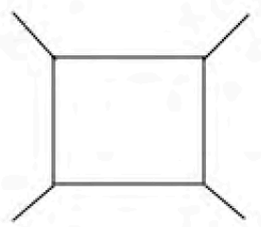
High school problem $gg \to gg$

- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$



$$\text{Vol}\,(1) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1} =$$

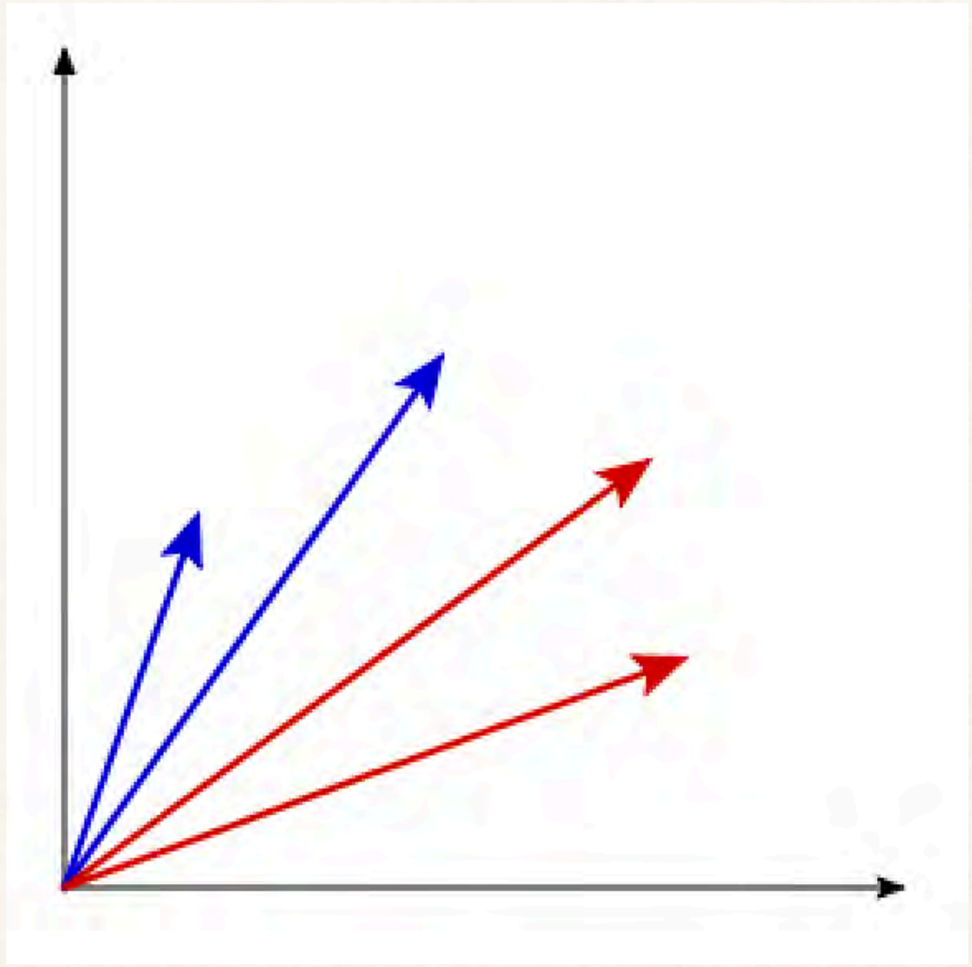


High school problem $gg \to gg$

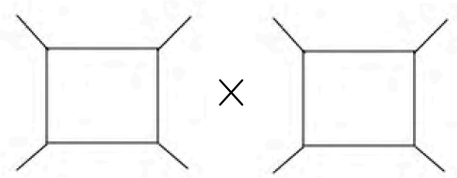
- Positive quadrant
- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$[\mathrm{Vol}\,(1)]^2 = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2} =$$



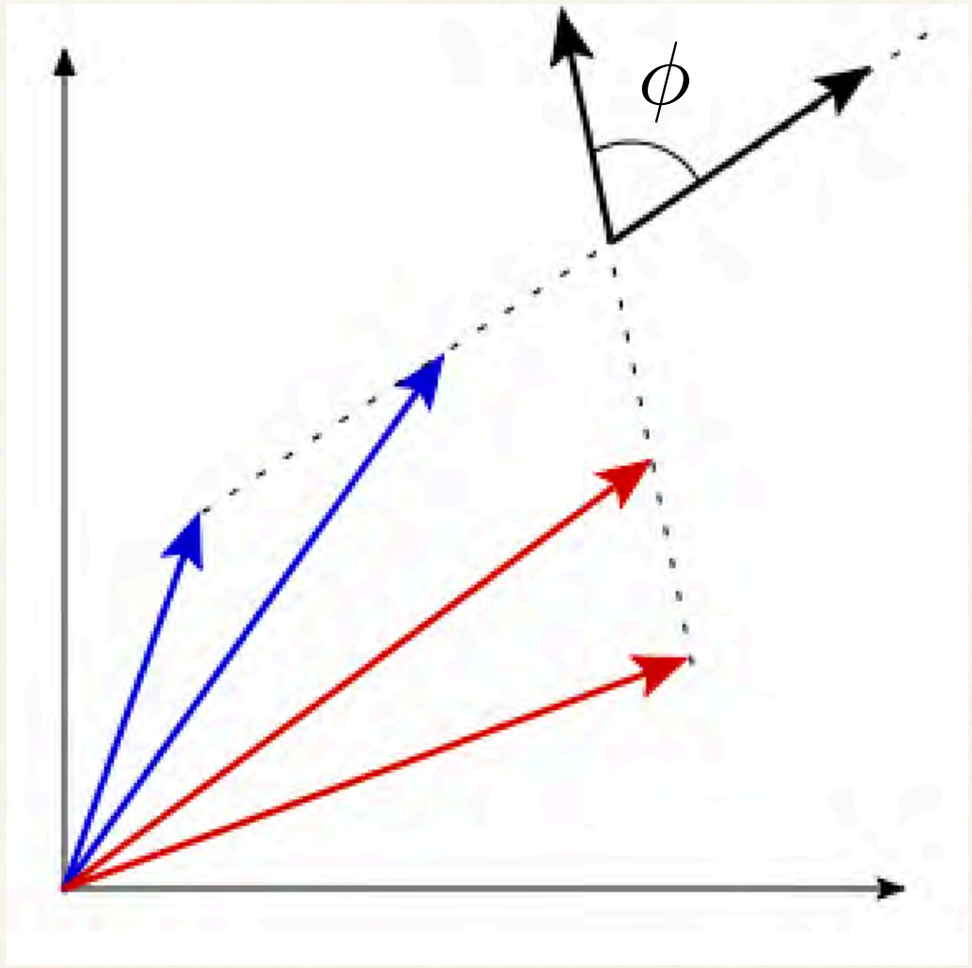
High school problem $gg \to gg$

- Positive quadrant

- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



- Impose: $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1) \leq 0 \qquad \phi > 90^o$

Subset of configurations allowed: triangulate

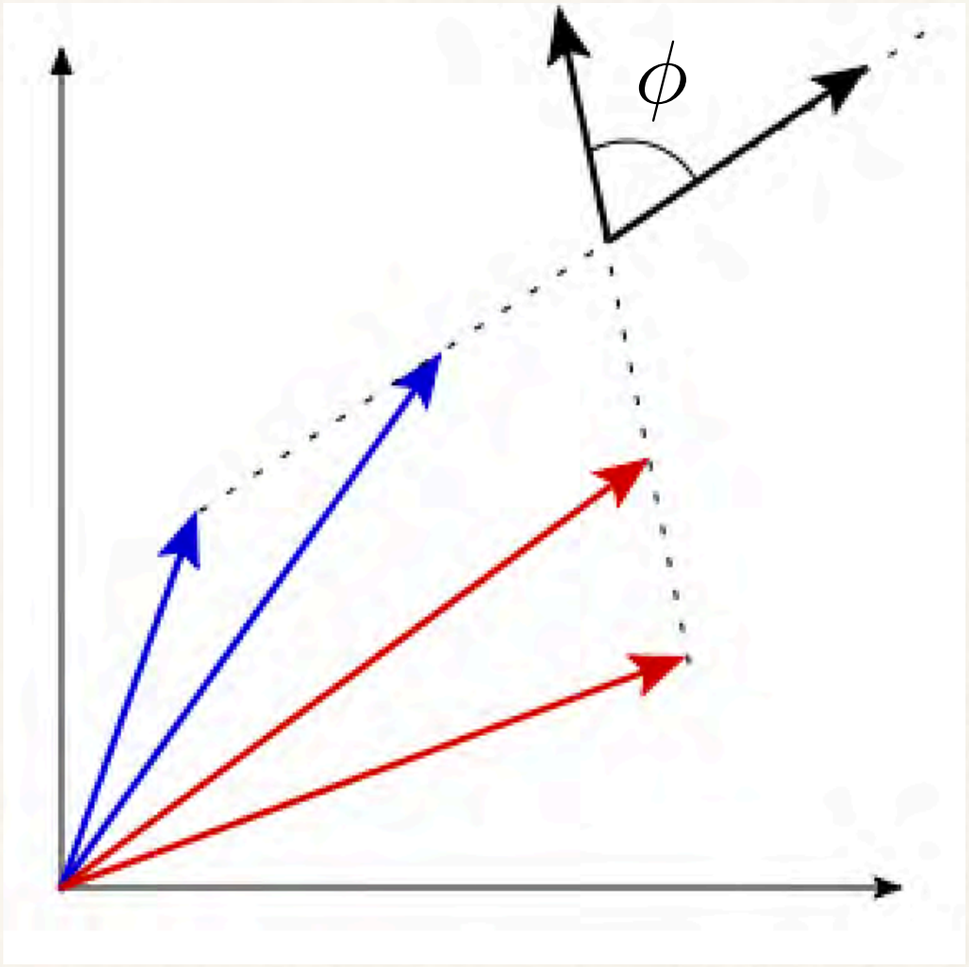
High school problem $gg \to gg$

- Positive quadrant

- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$$\text{Vol}\,(2) = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}\frac{dz_2}{z_2}\frac{dw_2}{w_2}\left[\frac{\vec{a}_1 \cdot \vec{b}_2 + \vec{a}_2 \cdot \vec{b}_1}{(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_2 - \vec{b}_1)}\right]$$

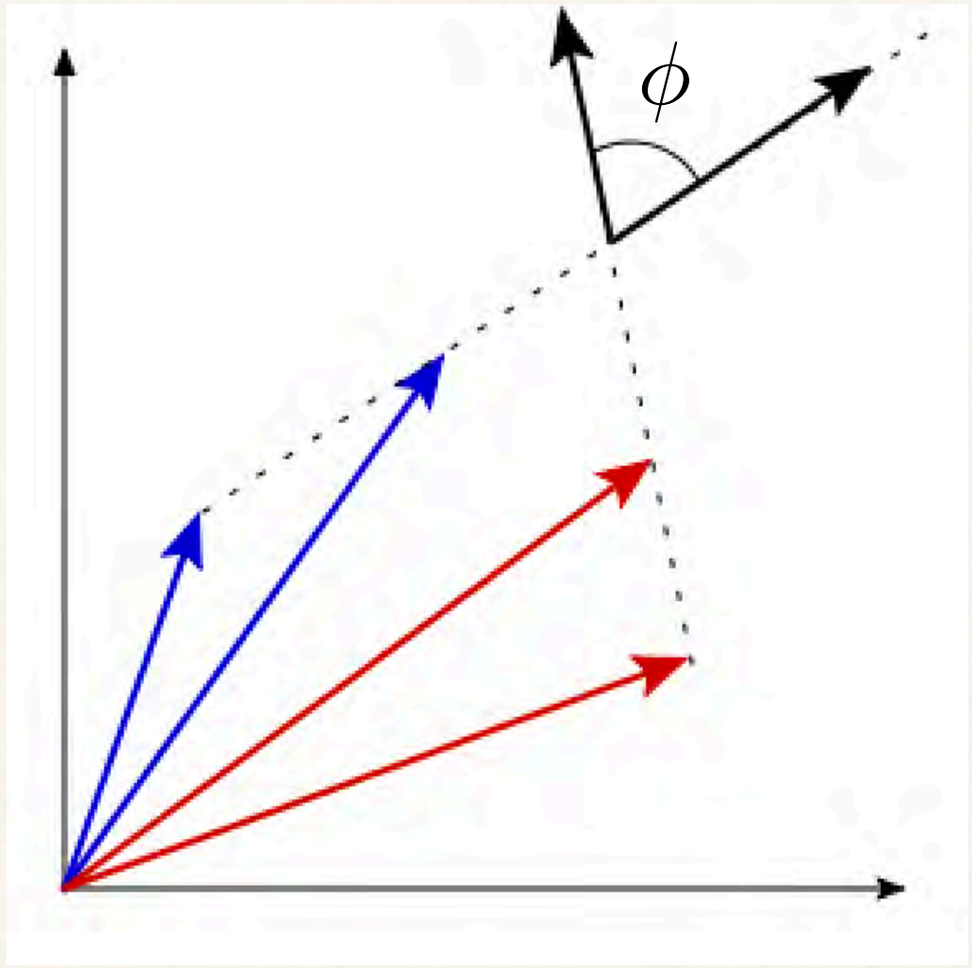
High school problem $gg \to gg$

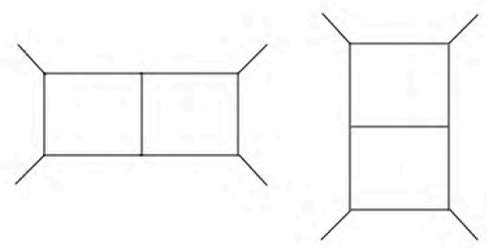
- Positive quadrant

- Vectors

$$\vec{a}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \vec{b}_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad \vec{b}_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$$



$\mathrm{Vol}\,(2) =$ 

High school problem $gg \to gg$

- Positive quadrant

- Vectors

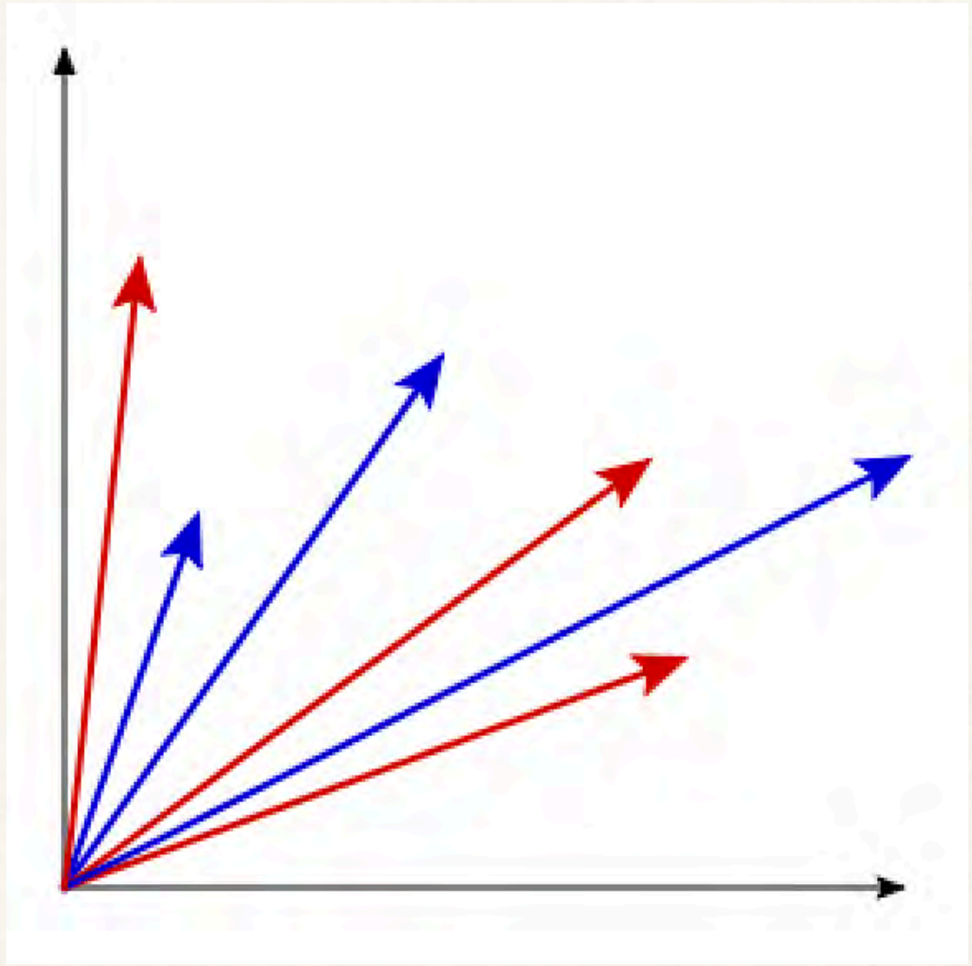
  $\vec{a}_1, \vec{a}_2, \vec{a}_3$ $\qquad$ $\vec{b}_1, \vec{b}_2, \vec{b}_3$

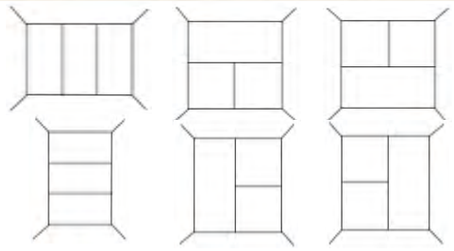
- Conditions

$$(\vec{a}_1 - \vec{a}_2) \cdot (\vec{b}_1 - \vec{b}_2) \leq 0$$

$$(\vec{a}_1 - \vec{a}_3) \cdot (\vec{b}_1 - \vec{b}_3) \leq 0$$

$$(\vec{a}_2 - \vec{a}_3) \cdot (\vec{b}_2 - \vec{b}_3) \leq 0$$



$\mathrm{Vol}\,(3) =$ 

High school problem $gg \to gg$

- Positive quadrant

- Vectors

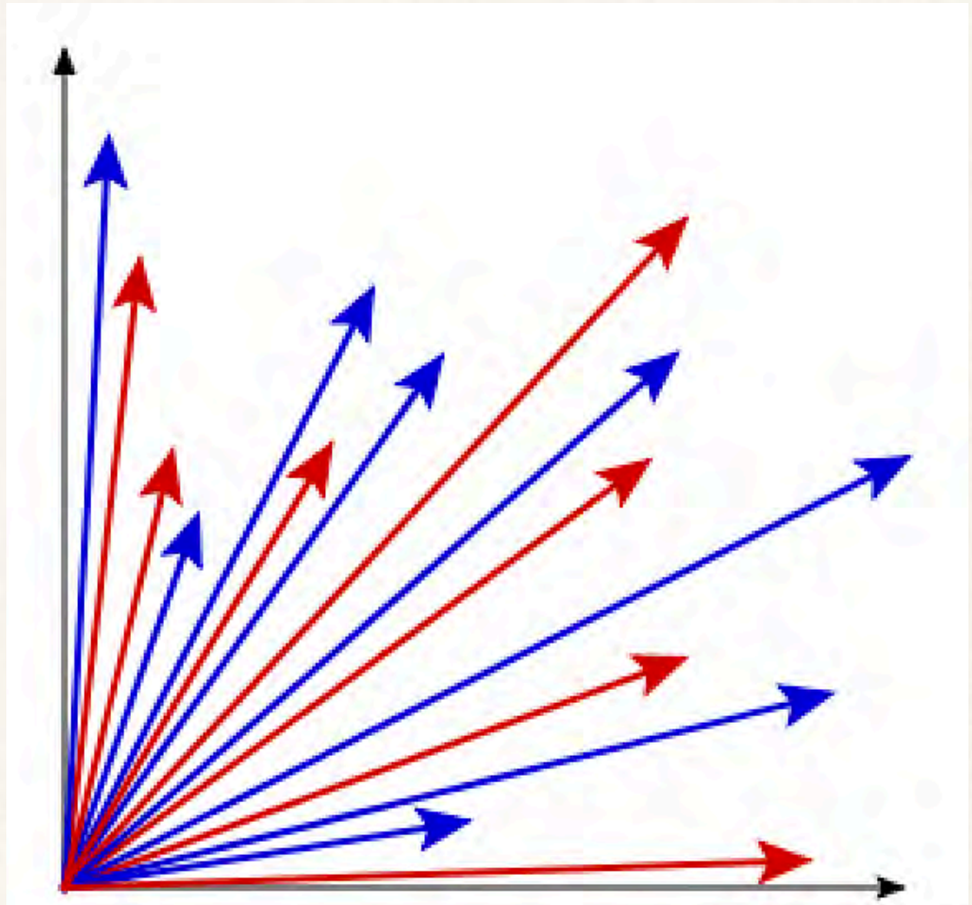
$$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_\ell \quad \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_\ell$$

- Conditions

$$(\vec{a}_i - \vec{a}_j) \cdot (\vec{b}_i - \vec{b}_j) \leq 0$$

for all pairs i, j

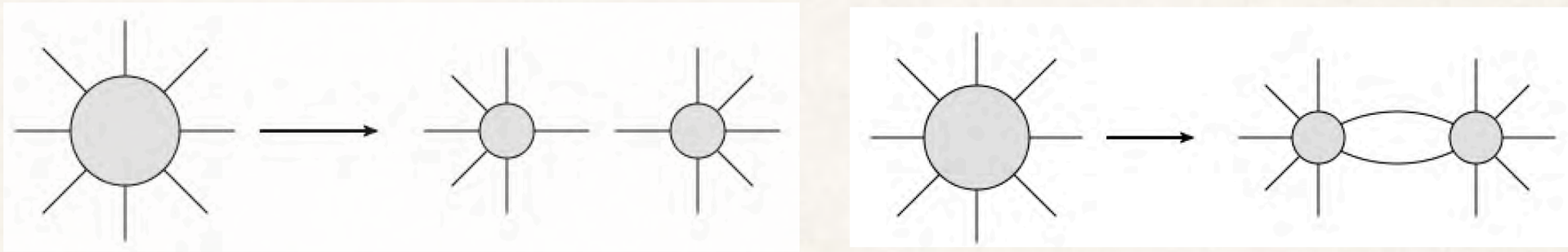
Let me know if you solve it!



$$\mathrm{Vol}\,(\ell) = \ldots\ldots$$

Why true?

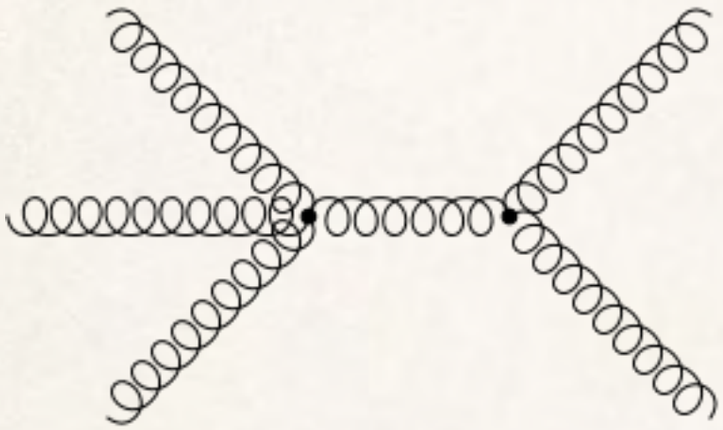
- No QFT proof
- It is correct: the result satisfies locality and unitarity

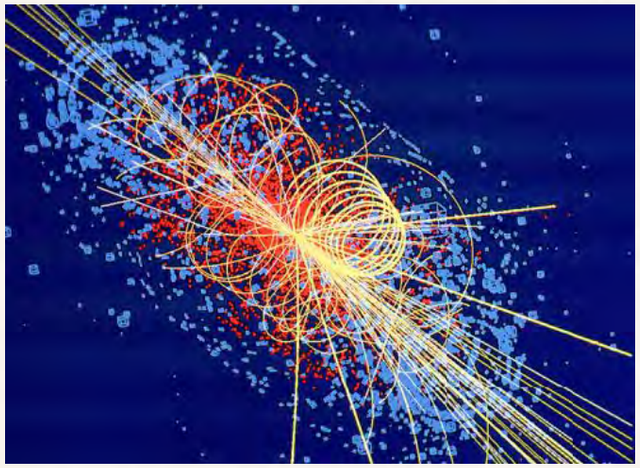


- Also check against results in the literature

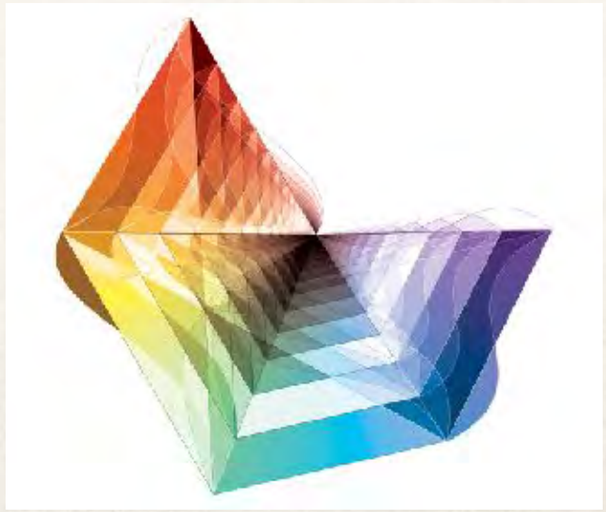
Physics vs geometry

- Dynamical particle interactions in 4-dimensions

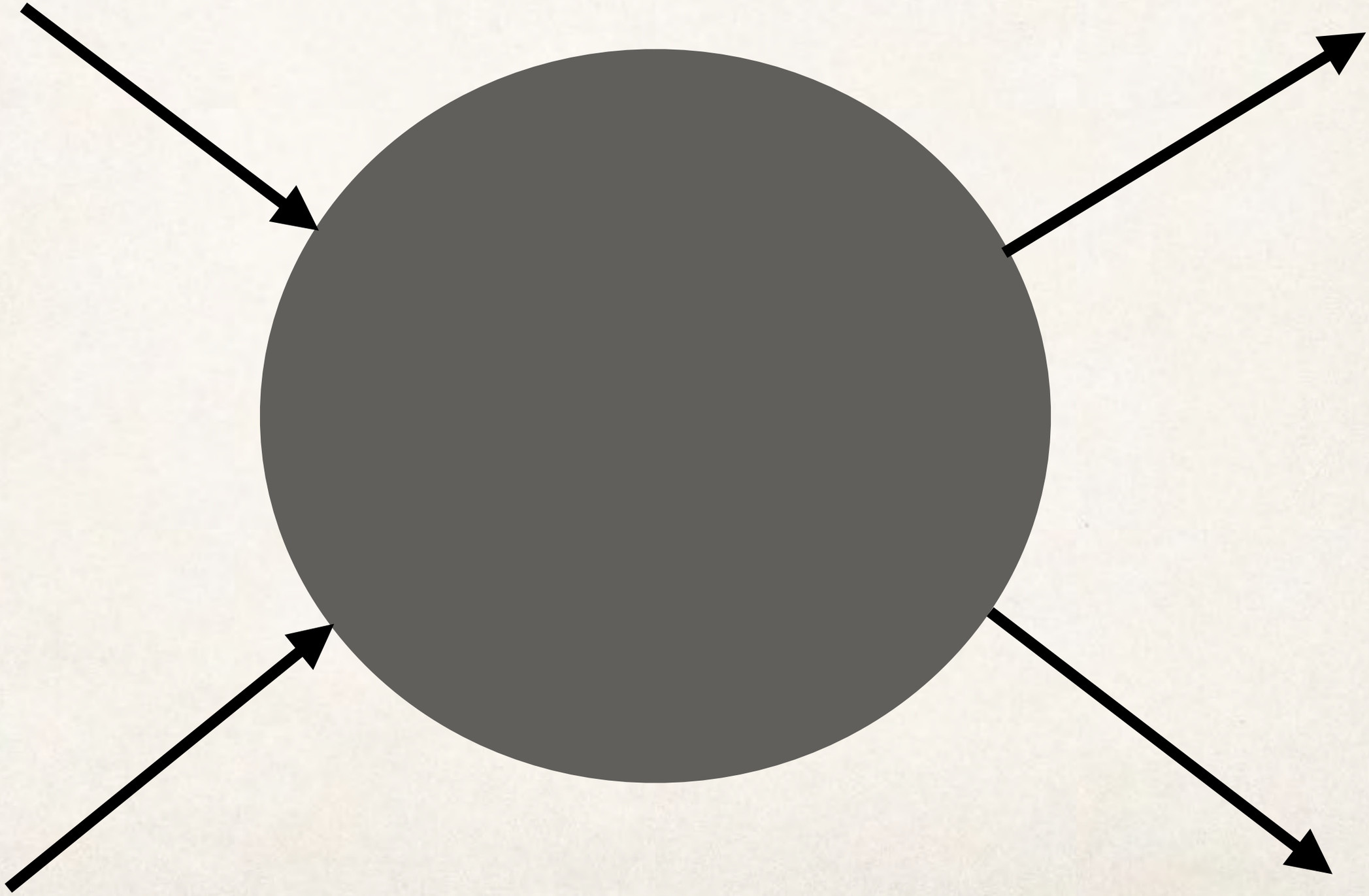




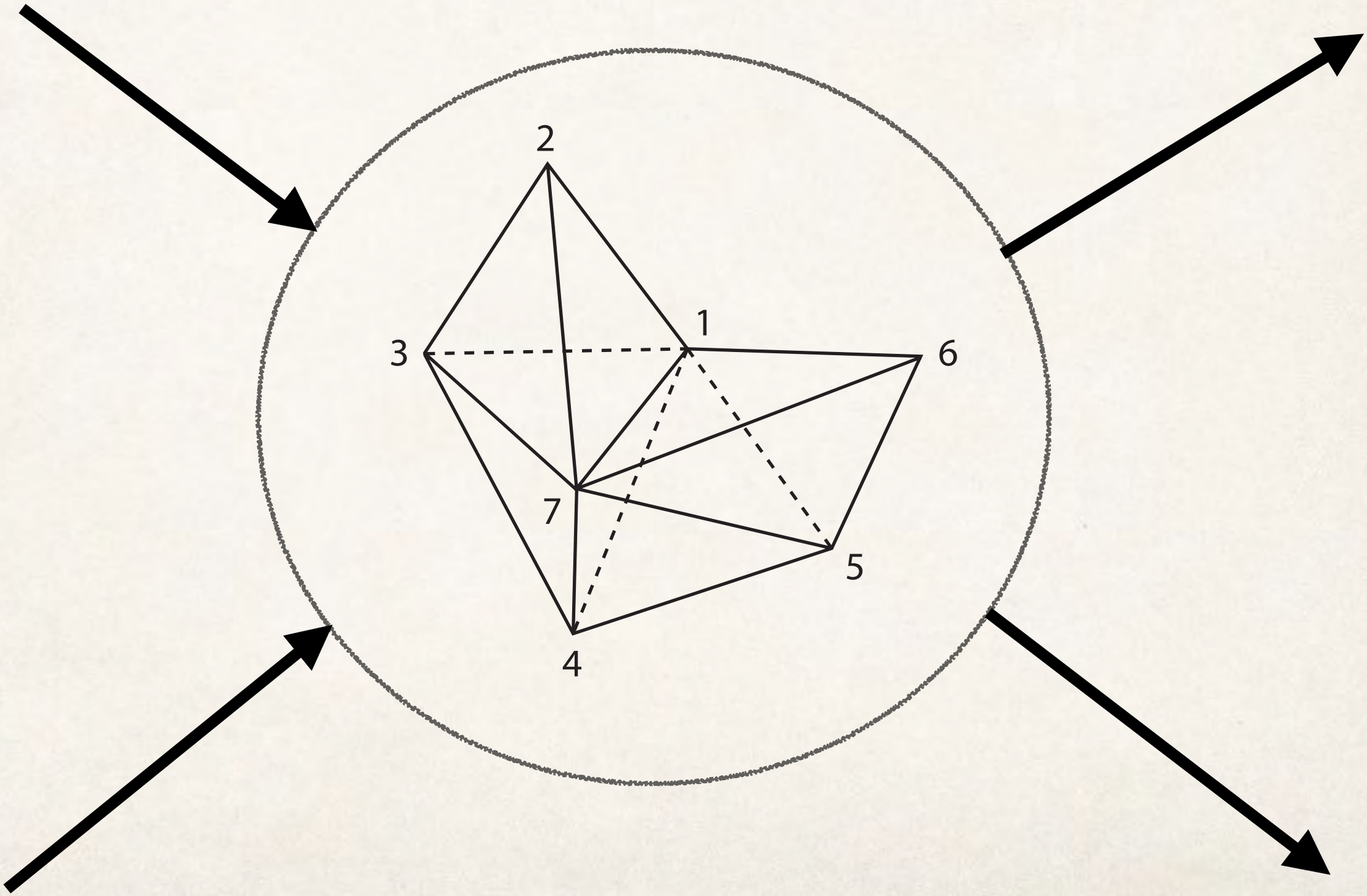
- Static geometry in high dimensional space



What is scattering amplitude?



What is scattering amplitude?

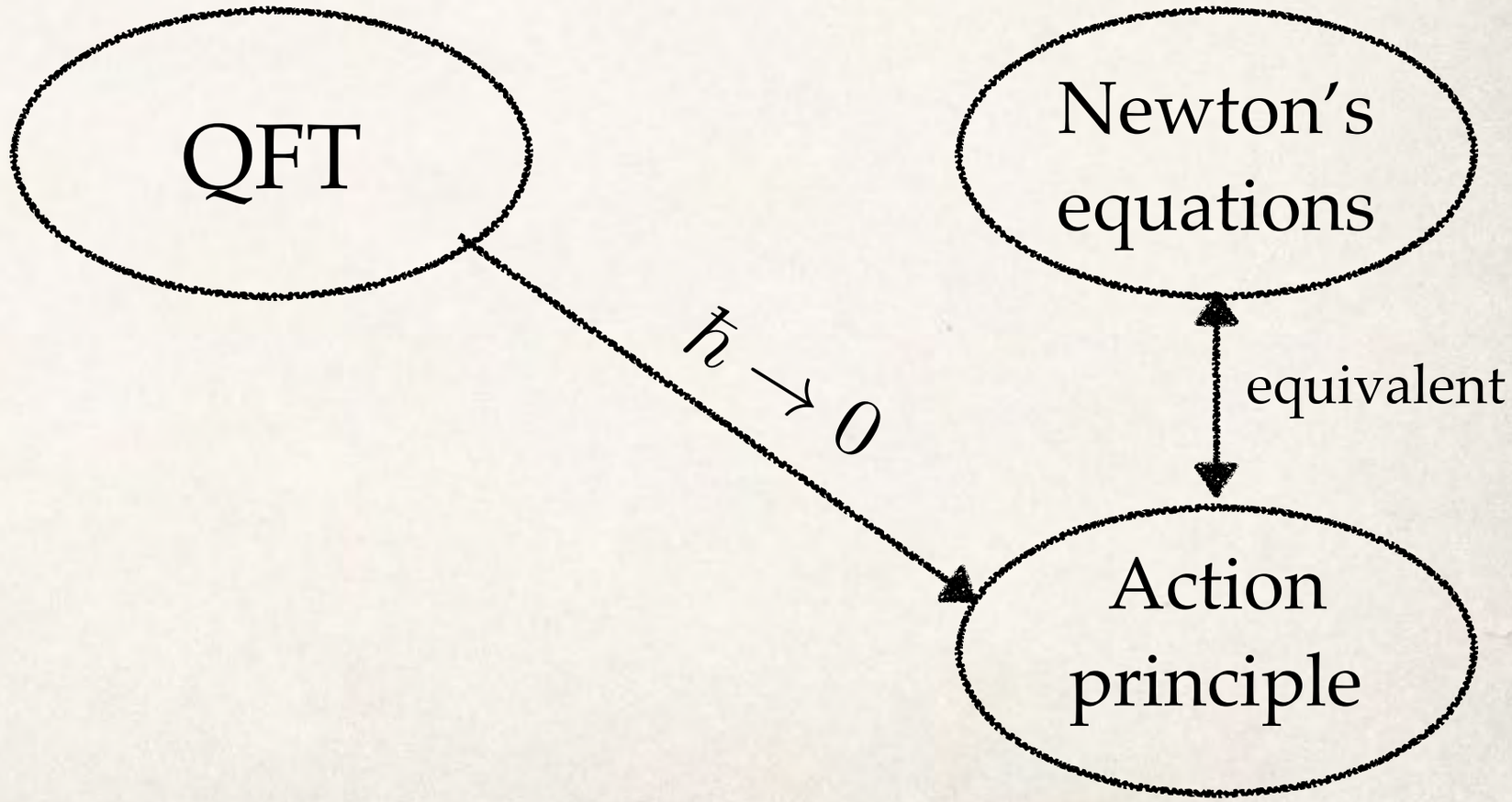


Step 1.1.1.

- It is very early to say if/how this can generalize
- Some encouraging news but more work needed
- Extend to other theories, beyond the integrand,….

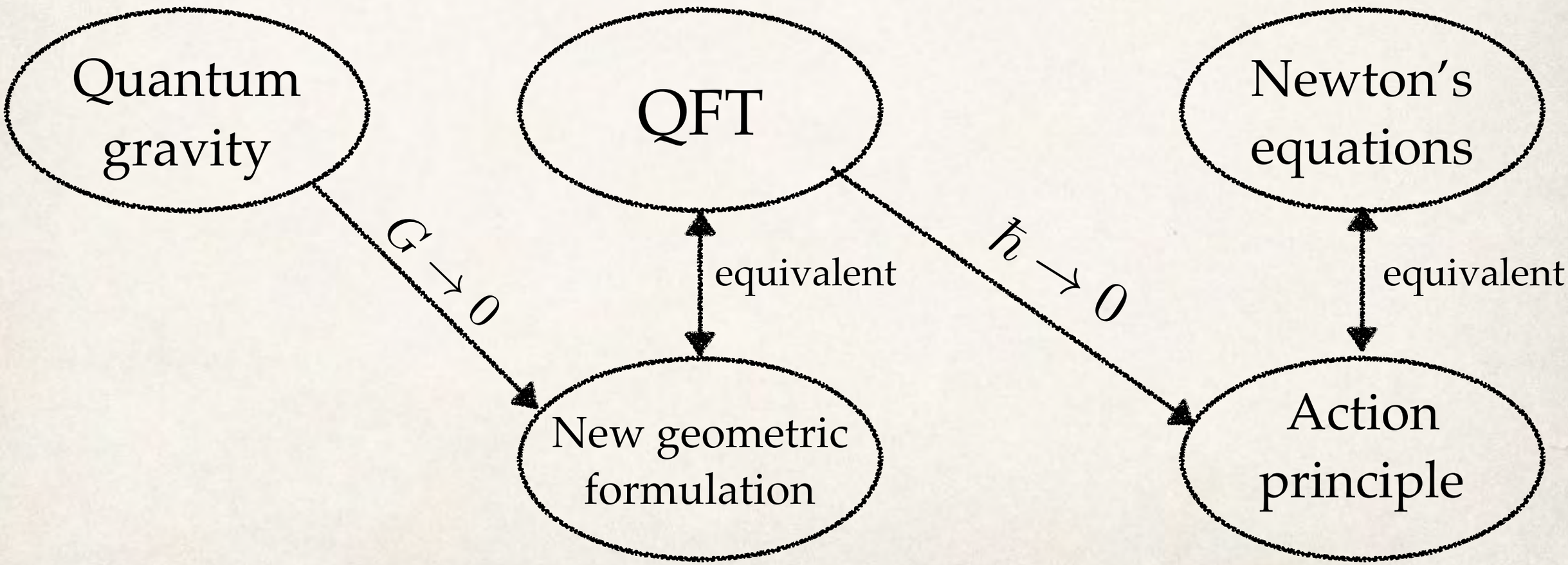
Fantasy

- Beyond understanding QFT better there is one more motivation



Fantasy

- Beyond understanding QFT better there is one more motivation



Amplitudes as a new field

- This is one of the directions in fast developing field

- Scattering equations, BCJ duality, string amplitudes, supergravity finiteness, hexagon bootstrap, cluster polylogarithms, worldsheet models, integration techniques, soft theorems, LHC calculations,…..

- Zeroth order problems open, many chances for young people to make big discoveries!

Amplitudes 2018 @ SLAC/Stanford

June 18-22

This event is the continuation of a sequence of conferences started in 2009:

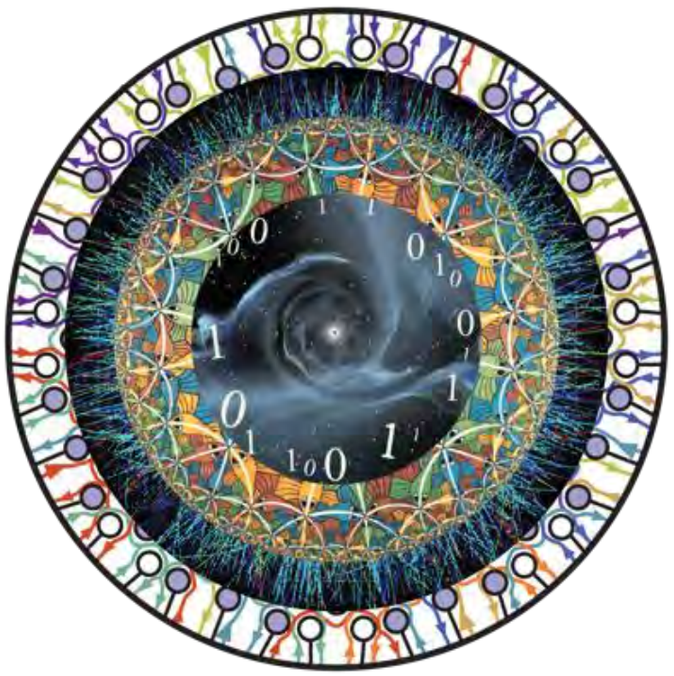
- 2009: IPPP Durham
- 2010: Queen Mary College, London
- 2011: University of Michigan, Ann Arbor
- 2012: DESY Hamburg
- 2013: Schloss Ringberg
- 2014: IPhT Saclay
- 2015: Zürich
- 2016: Nordita, Stockholm



School at QMAP (Center for Quantum Mathematics & Physics) @ UC Davis
June 11-15, 2018







SLAC NATIONAL ACCELERATOR LABORATORY

conf.slac.stanford.edu/amplitudes/

Thank you!